Infimal convolution of data discrepancies for mixed noise removal

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Abstract
We consider the problem of image denoising in the presence of noise whose statistical properties
are a combination of two different distributions. We focus on noise distributions frequently considered
in applications, such as salt & pepper and Gaussian, and Gaussian and Poisson noise mixtures. We
derive a variational image denoising model that features a total variation regularisation term and
a data discrepancy encoding the mixed noise as an infimal convolution of discrepancy terms of the
single-noise distributions. We give a statistical derivation of this model by joint Maximum A-Posteriori
(MAP) estimation, and discuss in particular its interpretation as the MAP of the so-called infinity
convolution of two noise distributions. Classical single-noise models are recovered asymptotically as
the weighting parameters go to infinity. The numerical solution of the model is computed using
second order Newton-type methods. Numerical results show the decomposition of the noise into its
constituting components. The paper is furnished with several numerical experiments and comparisons
with other methods dealing with the mixed noise case are shown.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ open and bounded with Lipschitz boundary and $f : \Omega \rightarrow \mathbb{R}$ be a given noisy image. The
image denoising problem can be formulated as the task of retrieving a denoised image $u : \Omega \rightarrow \mathbb{R}$ from $f$. In its
general expression, assuming no blurring effect on the observed image, the denoising inverse
problem assumes the following form:

$$\text{find } u \text{ s.t. } f = T(u),$$

where $T$ models the degradation process generating random noise that follows some statistical distribu-
tion. Due to the general ill-posedness of (1.1), such problem is often regularised and reformulated as the
following minimisation problem

$$\text{find } u \text{ s.t. } u \in \arg\min_{v \in V} \left\{ J(v) := R(v) + \lambda \Phi(v, f) \right\},$$

where now images are elements in suitable function spaces and the reconstructed image is computed as
a minimiser $u$ of the energy functional $J$ in the Banach space $V$ which is the sum of two different terms:
the regularising energy $R$ encoding a-priori information on the desired image $u$ and the data fidelity
function $\Phi$ modelling the relation between the data $f$ and the reconstructed image $u$. The positive
weighting parameter $\lambda$ balances the action of the regularisation against the trust in the data. In this
paper we consider the Total Variation (TV) as image regulariser

$$R(u) = |Du|(\Omega) = \sup_{\xi \in C^0_c(\Omega,\mathbb{R}^2)} \int u \text{ div}(\xi) \ dx,$$

which is a popular choice since the seminal works of Rudin, Osher and Fatemi [44], Chambolle and
Lions [10] and Vese [50] due to its edge-preserving and smoothing properties. Under this choice, the
minimisation problem (1.2) is formulated in a subspace of $BV(\Omega)$, the space of functions of bounded variation (see [1]).

Having fixed the regularisation term, depending on the specific application considered, several choices for the fidelity term $\Phi$ and the balancing parameter $\lambda$ can be made. The correct mathematical modelling of the data fidelity term $\Phi$ in (1.2) is crucial for the design of an image reconstruction model fitting appropriately the given data. The choice of $\Phi$ is typically driven by statistical considerations on the noise in the data $f$ in (1.1), cf. [49]. Namely, the general model (1.2) is derived as the MAP estimate of the likelihood distribution. In the simplest scenario, the noise is assumed to be an additive random variable $w$, Gaussian-distributed with zero mean and variance $\sigma^2$ determining the noise intensity. Gaussian noise is often used, for instance, to model the noise statistics in many medical imaging applications as a simple approximation of more complicated noise models. Other additive noise distributions, such as the Laplace distribution can alternatively be considered. Another possibility – which is appropriate for modelling transmission errors affecting only a percentage of the pixels in the image – is to consider a type of noise where the intensity value of only a fraction of pixels in the image is switched to either the maximum/minimum value of its dynamic range or to a random value within it with positive probability. This type of noise is called “salt & pepper” noise or “impulse” noise, respectively. In some other cases a different, signal-dependent property is assumed to conform to the actual physical application considered (for instance for astronomical and microscopy imaging applications). In this case, a Poisson distribution of the noise can be used.

When Gaussian noise is assumed, an $L^2$-type data fidelity
\[ \Phi_{Gauss}(u, f) := \frac{1}{2} \int_\Omega (u - f)^2 \, dx, \] (1.4)
can be derived for $f \in L^2(\Omega)$ as the MAP estimate of the Gaussian likelihood function [44] [16]. Similarly, in the case of additive Laplace noise, the statistically consistent data fidelity term for $f \in L^1(\Omega)$ reads
\[ \Phi_{Imp}(u, f) := \int_\Omega |u - f| \, dx, \] (1.5)
see, e.g. [39] [4]. Note that the same data fidelity is considered in [40] [26] to model the sparse structure of the salt & pepper and random noise distributions. Variational models where a Poisson noise distribution is assumed are often approximated by weighted-Gaussian distributions through variance-stabilising techniques [48] [9]. In [47] [46] a statistically-consistent analytical modelling is derived: for $f \in L^\infty(\Omega)$ the resulting data fidelity term is a Kullback-Leibler-type functional $\Phi$ of the form
\[ \Phi_{Pois}(u, f) := \int_\Omega (u - f \log u) \, dx. \] (1.6)

As a result of different image acquisition and transmission faults, in several applications the observed image is frequently corrupted by a mixture of noise statistics. Mixed noise distributions can be observed when faults in the acquisition of the image are combined with transmission errors to the receiving sensors. In this case, the modelling of $\Phi$ in (1.2) shall encode a combination of salt & pepper and Gaussian noise. In other applications, specific tools (such as fluorescence and/or high-energy beams) are used to generate the signal before it is actually acquired. This process is typical in microscopy and astronomy and may result in a mixture of a Poisson and Gaussian noise, see e.g. [45].

From a mathematical point of view, the presence of multiple noise distributions has been modelled in different ways as a combination of the data fidelity terms (1.4), (1.5) and (1.6). In [30], for instance, a combined model with $L^1 + L^2$ data fidelity and TV regularisation is considered for joint impulsive and Gaussian noise removal. There, the image is spatially decomposed into two disjoint domains in which the image is separately corrupted by pure salt & pepper and Gaussian noise, respectively. A two-phase approach is considered in [11] where two sequential steps with $L^1$ and $L^2$ data fidelity are performed to remove the impulsive and the Gaussian component of the mixed noise, respectively. In [25] [31] a framelet-based approach combining $\ell^1$ and $\ell^2$ data fidelities is considered for blind inpainting problems and mixed noise removal in a discrete framework. Mixtures of Gaussian and Poisson noise have also been considered. In [34] [33], for instance, the exact log-likelihood estimator of the mixed noise model is derived and its numerical solution is computed via a convergent primal-dual splitting algorithm. However, the
approximation of the infinite sum appearing in the data discrepancy considered badly affects efficiency and reflects in large computational times. A similar model was considered in [5] where a scaled gradient semi-convergent algorithm was used to solve the combined model. In [27] the discrete-continuous nature of the model (due to the different support of the Poisson-Gaussian distribution on the set of natural and real numbers, respectively) is approximated by an additive model, using homomorphic variance-stabilising transformations and weighted-$L^2$ approximations. In [37] a Gaussian-Poisson model similar to the one we consider in this paper is derived in a discrete setting. A more recent approach featuring a linear combination of different data fidelities of the type (1.4), (1.5) and (1.6) has been considered in [36] for a combination of Gaussian and impulse noise and in [22, 13, 12] in the context of bilevel learning approaches for the optimisation of denoising models.

In this work, we present a variational model for removing mixed noise from images, discussing in detail both its statistical origin and its analysis in function spaces. Our model is based on an infimal convolution modelling of the data discrepancies (1.4), (1.5) and (1.6) and is derived as the MAP estimator of the posterior distribution with a joint likelihood of two noise distributions. The derived model further connects with the notion of infinity convolution of probability densities, a standard notion in convex analysis. A unified analysis of the model in function spaces shows its well-posedness and an efficient Semi-Smooth Newton (SSN) method is derived for its numerical realisation, which makes it amenable for the embedding in a bilevel learning approach for its optimal design, see [22, 12]. Finally, our thorough statistical derivation and well-posedness analysis builds a bridge with related approaches considered for mixed noise removal in [25, 51, 37] where no statistical consistency property nor theoretical guarantee of their validity is given.

We will derive the combined model in a discrete dimensional setting and use this as a motivation for the subsequent analysis infinite dimensional function space framework. With a little abuse of terminology, we will refer to “noise components” to denote elements of such spaces associated to the impulsive, Gaussian and Poisson components of the model, as appropriate. However, the rigorous definition of noise in an infinite dimensional setting is a delicate matter which is beyond the purposes of this work.

In what follows, we fix the regularisation term in (1.2) to be the TV energy (1.3). This is a toy example and extensions to higher-order regularisations such as TV-TV² [42] or TGV [7] can be considered as well (this will be the topic of a forthcoming paper [14]). We will refer to our model as TV-IC to highlight the infimal convolution combination of data fidelities. This should not be confused with the ICTV model [16, 32] where the infimal-convolution operation is used to combine TV regularisation with second-order regularisation terms.

1.1 The discrete reference models

In the following, we consider two exemplar problems which extend (1.1) to the case of multiple noise distributions. For $u$ and $f$ in $\mathbb{R}^n$, we consider the following problem encoding noise mixtures

$$\text{find } u \text{ such that } f = \mathcal{T}(u) + w.$$  (1.7)

Similarly in (1.1), here $\mathcal{T}$ models a general noising process possibly depending on $u$ in a non-linear way, while $w$ is an additive, Gaussian-distributed, noise component independent of $u$. We will focus in the following on two particular cases of (1.7).

Salt & pepper and Gaussian noise We first suppose that the observed noisy image $f$ is corrupted by a mixture of salt & pepper and Gaussian noise, i.e. we consider the following instance of the noising model (1.7),

$$f = (1 - s)u + sc + w$$  (1.8)

where, following [13] Section 1.2.2, the salt & pepper component is modelled by considering two independent random fields $s$ and $c$ defined, for every pixel $x \in \Omega$ defined as

$$c(x) = \begin{cases} 0, & \text{with probability } 1/2 \\ 1, & \text{with probability } 1/2 \end{cases}, \quad s(x) = \begin{cases} 0, & \text{with probability } p \\ 1, & \text{with probability } 1 - p \end{cases}$$
Following [33] Section 2.2 and [39], we will approximate the nonlinear model (1.8) by the following additive model:

\[
f = u + v + w,
\]

where \(v\) is now the realisation of a Laplace distributed random variable, independent of \(u\). This approximation will be made more clear in Section 3.

**Poisson and Gaussian noise** We consider also a mixture of noise distributions where a signal-dependent component is combined with an additive Gaussian noise component. In particular, we will focus on a Poisson distributed component \(z\) having \(u\) as Poisson parameter and consider the noising model (1.7) with

\[
f = z + w, \quad \text{where } z \sim \text{Pois}(u).
\]

**Organisation of the paper** In Section 2 we present the infimal-convolution data fidelity terms proposed for the reference models of Section 1.1 and show that they are well-defined in function spaces. Statistical motivations for the TV-IC denoising model are given in Section 3 where we make precise how they fit into the general model above. For the model (1.9), the expression of \(\Phi^{\lambda_2}\) will be specified in the following.

\[
\Phi^{\lambda_2}(u,f) := \inf_{v \in L^2(\Omega) \cap B} \left\{ \mathcal{F}^{\lambda_1,\lambda_2}(f,u,v) := \lambda_1 \Phi_1(v) + \lambda_2 \Phi_2(u,f-v) \right\},
\]

for two data fidelity functions \(\Phi_1\) and \(\Phi_2\) defined in \(L^2(\Omega) \cap B\). The size of the parameters \(\lambda_1\) and \(\lambda_2\) in \(\Phi^{\lambda_1,\lambda_2}\) weights the trust in the data against the smoothing effect of TV regularisation as well as the fitting with the single noise models \(\Phi_1\) and \(\Phi_2\).

We now consider the two particular noise combinations introduced in Section 1.1 and demonstrate how they fit into the general model above.

### 2.1 Salt & pepper-Gaussian fidelity

For the model (1.9), the expression of \(\Phi^{\lambda_1,\lambda_2}\) in (TV-ICb) specifies as follows. Recalling (1.4) and (1.5) and for \(f \in L^2(\Omega)\), we set \(\Phi_1(v) = \Phi_{\text{Imp}}(v,0) = \|v\|_{L^1(\Omega)}\) for the impulsive noise component and \(\Phi_2(u,f-v) = \Phi_{\text{Gauss}}(u,f-v) = \frac{1}{2} \|f - v - u\|_{L^2(\Omega)}^2\) for the Gaussian noise component. In this case \(\mathcal{A} = \mathcal{B} = L^2(\Omega)\) and \(\Phi^{\lambda_1,\lambda_2}\) reads

\[
\Phi^{\lambda_1,\lambda_2}(u,f) = \inf_{v \in L^2(\Omega)} \left\{ \mathcal{F}^{\lambda_1,\lambda_2}(f,u,v) := \lambda_1 \|v\|_{L^1(\Omega)} + \lambda_2 \frac{1}{2} \|f - v - u\|_{L^2(\Omega)}^2 \right\},
\]
Since the set $\Omega$ is bounded, $L^2(\Omega) \subset L^1(\Omega)$ and both terms in (2.2) are well-defined.

**Remark 2.1.** Note that (2.2) can be rewritten as the Moreau envelope $M_\gamma \Phi_1$ of the $L^1$-norm in terms of the ratio between the parameters $\gamma := \lambda_1/\lambda_2$. Namely, we have:

$$\Phi_{\lambda_1,\lambda_2}(u, f) = \lambda_1 M_\gamma \Phi_1(f - u) = \lambda_1 \left( \Phi_1(\text{Prox}_\gamma \Phi_1(f - u)) + \frac{1}{\gamma} \Phi_2(u, f - \text{Prox}_\gamma \Phi_1(f - u)) \right).$$

Here, $\text{Prox}_\gamma \Phi_1$ denotes the proximal map of the function $\gamma \Phi_1$ (see [3, Section 12.4]). Moreover, via a similar argument as in [10, Proposition 3.3], one can show that $\Phi_{\lambda_1,\lambda_2}(u, f)$ is equivalent to the Huber regularisation of the $L^1$ norm multiplied by $\lambda_1$.

The following proposition asserts that the minimisation problem (2.2) is well-posed.

**Proposition 2.2.** Let $f \in L^2(\Omega)$, $u \in BV(\Omega) \subset L^2(\Omega)$ and $\lambda_1, \lambda_2 > 0$. Then the minimum in the minimisation problem (2.2) is uniquely attained.

**Proof.** The proof of this proposition is based on the use of standard tools of calculus of variations. We report it in the supplementary material. $\square$

### 2.2 Gaussian-Poisson fidelity

For the case of mixed Gaussian and Poisson noise (1.10) some additional assumptions need to be specified. We assume that $f \in L^\infty(\Omega)$ and that the data fidelities $\Phi_1$ and $\Phi_2$ are chosen as $\Phi_1(v) = \Phi_{\text{Gauss}}(v, 0) = \frac{1}{2} \| v \|_{L^2(\Omega)}^2$ and as

$$\Phi_2(u, f - v) = D_{KL}(f - v, u) = \int_\Omega \left( u - (f - v) + (f - v) \log \left( \frac{f - v}{u} \right) \right) \, d\mu, \quad (2.3)$$

the Kullback-Leibler (KL) divergence functional between $u$ and the “residual” $f - v$. We refer the reader to the supplementary material (Section 2) where more properties of the KL functional are recalled. The variational fidelity $\Phi_{\lambda_1,\lambda_2}$ in (TV-ICb) reads

$$\Phi_{\lambda_1,\lambda_2}(u, f) = \inf_{v \in L^2(\Omega) \cap \mathcal{B}} \left\{ \mathcal{F}^\gamma_{\lambda_1,\lambda_2}(f, u, v) = \frac{\lambda_1}{2} \| v \|_{L^2(\Omega)}^2 + \lambda_2 D_{KL}(f - v, u) \right\} \quad (2.4)$$

under the following choice of the admissible sets:

$$\mathcal{A} = \{ u \in L^1(\Omega), \log u \in L^1(\Omega) \} \quad \text{and} \quad \mathcal{B} = \{ v \in L^2(\Omega) : v \leq f \ \text{a.e.} \}. \quad (2.5)$$

Observe that for $u \in \mathcal{A}$ we have $u \geq 0$ almost everywhere. Together with the definition of $\mathcal{B}$, this ensures that the $D_{KL}$ functional is well defined (using the convention $0 \log 0 = 0$).

**Remark 2.3.** Note that the data discrepancy (2.3) differs from $\Phi_{\text{Pois}}(u, f)$ defined in (1.6) and used, for instance, in [40, 8, 3]. In those works, this choice is motivated through Bayesian derivation and MAP estimation and the terms that do not depend on the de-noised image $u$ are neglected since they are not part of the optimisation argument. However, in our combined modelling those quantities have to be taken into account to incorporate the additional variable $v$ encoding the Gaussian noise component.

Similarly as in Proposition 2.2 we guarantee that the minimisation problem (2.4) is well posed in the following proposition.

**Proposition 2.4.** Let $f \in L^\infty(\Omega)$, $u \in BV(\Omega) \cap \mathcal{A}$ and $\lambda_1, \lambda_2 > 0$. Let $\mathcal{A}$ and $\mathcal{B}$ be as in (2.5). Then, the minimum in the minimisation problem (2.4) is uniquely attained.

**Proof.** We report the proof in Section 1 of the supplementary material. It is based on standard tools of calculus of variations and on the properties of Kullback-Leibler divergence recalled in Section 2 of the supplementary material. $\square$
3 Statistical derivation and interpretation

In this section we want to give a statistical motivation for the choice of the IC data fidelity term (TV-ICb).

We do this by switching from the continuous setting with \( f \) in an infinite dimensional function space, to a discrete setting, i.e. considering a discrete domain \( \Omega := \{x_1, \ldots, x_K\} \times \{y_1, \ldots, y_L\} \) with cardinality \( |\Omega| = KL = M \) and the noisy datum \( f = (f_i)_{i=1}^M \) as an element of \( \mathbb{R}^M \). In this setting, we will show how the IC data fidelity (TV-ICb) can be derived in terms of a joint MAP estimate for \( u \) and \( v \) (Section 3.1) and by considering a modified likelihood for the mixed noise distribution – which consists of an infinity convolution of the two noise distributions – and computing its MAP estimate with respect to \( u \) (Section 3.2).

3.1 Joint maximum-a-posteriori estimation

A standard approach used to derive statistically consistent variational regularisation methods for inverse problems is the Bayesian approach \([18, 49]\). In this framework, the problem is formulated in terms of the maximisation of the posterior probability density \( P(u|f) \), i.e. the probability of observing the desired de-noised image \( u \) given the noisy image \( f \). This approach is commonly known as Maximum A Posteriori (MAP) estimation and relies on the simple application of the Bayes’ rule

\[
P(u|f) = \frac{P(f|u)P(u)}{P(f)},
\]

by which maximising \( P(u|f) \) for \( u \) translates in maximising the ratio on the right hand side of (3.1).

Classically, the probability density \( P(u) \) is called prior probability density since it encodes statistical \textit{a priori} information on \( u \). Frequently, a Gibbs model of the form

\[
P(u) = e^{-\alpha R(u)}, \quad \alpha > 0,
\]

where \( R(u) \) is a convex regularising energy functional is assumed. In what follows, we take \( R \) to be a discretisation of the total variation \((1.3)\), i.e. for each \( k, l \) pixel-coordinate, or, equivalently, reordering within a one-dimensional vector with index \( i \) (e.g., \( i = (l-1) \ast K + k \)), for \( u = (u_i)_{i=1}^M \in \mathbb{R}^M \), we have

\[
R(u) = \|\nabla u\|_{2,1} = \sum_{i=1}^M |\nabla u_i| = \sum_{k=1, \ldots, K, \ell=1, \ldots, L} \sqrt{(u_{k+1,l} - u_{k,l})^2 + (u_{k,l+1} - u_{k,l})^2},
\]

where here \( \nabla \) denotes the forward-difference approximation of the two-dimensional gradient. The conditional probability density \( P(f|u) \), also called likelihood function, relates to the statistical assumptions on the noise corrupting the data \( f \). Its expression depends on the probability distribution of the noise assumed on the data. The denominator of (3.1) plays the role of a normalisation constant for \( P(u|f) \) to be a probability density.

Maximising (3.1) in our setting with a Gibbs prior is equivalent to minimising the negative logarithm of \( P(u|f) \). Thus, the MAP estimation problem equivalently corresponds to the problem of finding

\[
u_{\text{MAP}} \in \arg \min_u \{-\log P(u|f)\} = \arg \min_u \{-\log P(f|u) + \alpha R(u)\},
\]

where the term \( \log P(f) \) has been neglected since it does not affect the minimisation over \( u \).

In what follows, we want to use MAP estimation to derive a discrete version of the TV-IC variational model (TV-ICA)–(TV-ICb) for the mixed noise scenario. From a Bayesian point of view, the general model (1.7) can be written in terms of random variables \( F \) and \( U \) (for which the noisy image \( f \) and the noise-free image \( u \) are realisations, respectively) as follows

\[
F = Z + W, \quad Z \sim P_Z^U, \quad U \sim P_U, \quad W \sim P_W := N(0, \sigma^2 I_d),
\]

where \( \sigma \) is the standard deviation of the Gaussian component. Here, \( F \) is the sum of the two random variables \( Z \) and \( W \). The former is distributed according to a probability distribution \( P_Z^U \) which may in general depend on \( U \) (for instance, \( U \) could be one of its parameters), whereas \( W \) is fixed to be a Gaussian
random variable independent of $U$ and $Z$, which is combined with $Z$ through an additive modelling. In (3.4) we have denoted by $P_U$ the prior Gibbs model (3.2) on $U$.

For our derivation we consider a joint MAP estimation for a pair of realisations $(u, w)$ of the corresponding random variables $U$ and $W$. Using Bayes’ rule and mutual independence between $W$ and $U$, we have:

$$
(\hat{u}, \hat{w}) = \arg \max_{(u, w)} P(u, w | f) = \arg \max_{(u, w)} P(f | u, w) P(u, w) = \arg \max_{(u, w)} P_Z^U(f - w) P_W(w) P_U(u) \quad (3.5)
$$

which holds for general noise distributions $P_Z^U$. However, in the particular case where $Z = U + V$ with $V$ being a random variable distributed as $V \sim P_V$ and independent of $U$ (in particular, when $U$ is not a parameter of the probability distribution of $Z$ and is combined additively with another random variable $V$), (3.4) becomes

$$
F = U + V + W. \quad (3.6)
$$

In this case $P_Z^U(z) = P_V(z - u)$, and (3.5) reduces to:

$$
(\hat{u}, \hat{w}) = \arg \max_{(u, w)} P_Z^U(f - w) P_W(w) P_U(u) = \arg \max_{(u, w)} P_U(f - u - w) P_W(w) P_U(u). \quad (3.7)
$$

After these general considerations, we specify the MAP derivation of the TV-IC model first for the additive mixture of Laplace and Gaussian noise distribution (1.9), and then for the mixed Gaussian-Poisson case (1.10).

**The additive case** Let us focus first on the additive model (3.6). Following [43], we will use the heavy-tailed Laplace distribution as an approximation of the one describing salt & pepper noise, compare with [39]. For each image pixel $i = 1, \ldots, M$, the discretised problem (3.6) reads

$$
\text{find } u_i \text{ s.t. } f_i = u_i + v_i + w_i, \quad (3.8)
$$

where $v_i$ and $w_i$ are realisations of two mutually independent random variables distributed with Laplace and Gaussian distribution, respectively. More precisely, the equation above describes the structure of the realisations $(f_1, \ldots, f_M)$ of the components of the random vector $F := (F_1, \ldots, F_M)$ in terms of the ones of the random vectors $U := (U_1, \ldots, U_M)$, $V := (V_1, \ldots, V_M)$ and $W := (W_1, \ldots, W_M)$. The quantities $v_i$ and $w_i$ are, for every $i = 1, \ldots, M$, independent realisations of identically distributed Laplace and Gaussian random variables $V_i$ and $W_i$, respectively, i.e. $V_i \sim \text{Lapl}(0, \tau) = P_{V_i}$ and $W_i \sim \mathcal{N}(0, \sigma^2) =: P_{W_i}$.

We recall the expression of Laplace probability density with parameter $\tau > 0$ and the Gaussian probability density with zero-mean and variance $\sigma^2$:

$$
P_{V_i}(s) = \frac{e^{-|s|/\tau}}{2\tau}, \quad P_{W_i}(s) = \frac{e^{-|s|^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}, \quad s \in \mathbb{R}. \quad (3.9)
$$

Let us now take the negative logarithm in the joint MAP estimate (3.7) for all the realisations. By independence and recalling the Gibbs model (3.2)- (3.3) for $P_U(u)$, we have:

$$
(\hat{u}, \hat{w}) = \arg \min_{u=(u_1, \ldots, u_M), \atop w=(w_1, \ldots, w_M)} - \log \left( P_V(f - u - w) P_W(w) P_U(u) \right)
$$

$$
= \arg \min_{u=(u_1, \ldots, u_M), \atop w=(w_1, \ldots, w_M)} \left\{ - \log \left( P_V(f - u - w) \right) - \log \left( P_W(w) \right) - \log \left( P_U(u) \right) \right\}
$$

$$
= \arg \min_{u=(u_1, \ldots, u_M), \atop w=(w_1, \ldots, w_M)} \left\{ - \log \left( \prod_{i=1}^{M} P_V(f_i - u_i - w_i) \right) - \log \left( \prod_{i=1}^{M} P_W(w_i) \right) + \alpha R(u) \right\} \quad (3.10)
$$

$$
= \arg \min_{u=(u_1, \ldots, u_M), \atop w=(w_1, \ldots, w_M)} \sum_{i=1}^{M} \left( \frac{|f_i - u_i - w_i|}{\tau} + \frac{|w_i|^2}{2\sigma^2} + \alpha |\nabla u_i| \right).
$$

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where the constant terms not affecting the minimisation over \( u \) and \( w \) have been neglected.

To pass from a discrete to a continuous expression of the model we follow [46]. We interpret the elements in the space \( \mathbb{R}^M \) as samples of functions defined on the whole continuous image domain \( \Omega \). For convenience, we use in the following the same notation to indicate the corresponding, continuous quantities. Introducing the indicator function

\[
\chi_{D_i}(x) = \begin{cases} 
1, & \text{if } x \in D_i, \\
0, & \text{else},
\end{cases}
\]

where \( D_i \) is the region in the image occupied by the \( i \)-th detector, we have that any discrete data \( h_i \) can be interpreted as the mean value of a function \( h \) over the region \( D_i \) as follows

\[
h_i = \int_{D_i} h(x) \, dx = \int_{\Omega} \chi_{D_i}(x) h(x) \, dx.
\]

In this way, we can then express (3.10) as the following continuous model

\[
\min_{u, w: \Omega \to \mathbb{R}} \int_{\Omega} \left( \frac{|f(x) - u(x) - w(x)|}{\tau} + \frac{|w(x)|^2}{2\sigma^2} + \alpha |\nabla u(x)| \right) \, d\mu(x)
\]

where \( d\mu(x) = \sum_{i=1}^M \chi_{D_i} \, dx \) with \( dx \) being the usual Lebesgue measure in \( \mathbb{R}^2 \). We observe that at this level, the function spaces (i.e. the regularity) where the minimisation problem is formulated still need to be specified. Defining \( \lambda_1 := 1/\alpha \tau \) and \( \lambda_2 := 1/\alpha \sigma^2 \) and replacing \( \int_{\Omega} |\nabla u(x)| \, dx \) by \( |Du|_1(\Omega) \), we derive from (3.21) the following variational model for noise removal of mixed Laplace and Gaussian noise

\[
\min_{w \in BV(\Omega) \atop u \in L^2(\Omega)} |Du|_1(\Omega) + \lambda_1 \|f - u - w\|_{L^1(\Omega)} + \frac{\lambda_2}{2} \|w\|_{L^2(\Omega)},
\]

where the function spaces for \( u \) and \( w \) have now been chosen so that all the terms are well defined. By a simple change of variables, we can equivalently write the model above as

\[
\min_{u \in BV(\Omega) \atop v \in L^2(\Omega)} \|Du\|_1(\Omega) + \lambda_1 \|v\|_{L^1(\Omega)} + \frac{\lambda_2}{2} \|f - u - v\|_{L^2(\Omega)}, \tag{3.11}
\]

where the minimisation is taken over the de-noised image \( u \) and the salt & pepper noise component \( v \).

The signal-dependent case We now consider (3.4) in the case when the non-Gaussian noise component \( Z \) follows a Poisson probability distribution with parameter \( U \). In this case, the model (3.4) can be formulated in a finite-dimensional setting as

\[
f_i = z_i + w_i, \quad \text{with } z_i \sim Poiss(u_i) \tag{3.12}
\]

at every pixel \( i = 1, \ldots, M \). The equation above describes the structure of the realisations of the random vector \( F = (F_1, \ldots, F_M) \) in terms of the sum of two mutually independent random vectors \( Z := (Z_1, \ldots, Z_M) \) and \( W := (W_1, \ldots, W_M) \), having as components independent and identically distributed Poisson and Gaussian random variables \( Z_i \sim Pois(u_i) \) and \( W_i \sim N(0, \sigma^2) \), respectively, for every \( i = 1, \ldots, M \). The values \( u_i \) are realisations of the random variables \( U_i \) and \( U \) is distributed according to the Gibbs model (3.2) introduced before.

The Poisson noise density with parameter \( u_i \) is defined as

\[
P_{Z_i}^U(z_i) = \frac{u_i^{z_i} e^{-u_i}}{z_i!} = \exp(-u_i + \log\left(\frac{u_i^{z_i}}{z_i!}\right)), \quad z_i \in \mathbb{R}, \tag{3.13}
\]

where the factorial function is classically extended to the whole real line by using the Gamma function.
Similarly as above, we now take the negative logarithm in the joint MAP estimate (3.7), for all the realisations. By independence, we have:

\[
(\hat{u}, \hat{w}) = \arg\min_{u=(u_1, \ldots, u_M), \ w=(w_1, \ldots, w_M)} - \log \left( P_Z^U(f - w) P_W(w) P_U(u) \right)
\]

\[
= \arg\min_{u=(u_1, \ldots, u_M), \ w=(w_1, \ldots, w_M)} \left\{ - \log \left( \prod_{i=1}^M P_Z^{U_i}(f_i - w_i) \right) - \log \left( \prod_{i=1}^M P_W(w_i) \right) + \alpha R(u) \right\}
\]

\[
= \arg\min_{u=(u_1, \ldots, u_M), \ w=(w_1, \ldots, w_M)} \sum_{i=1}^M \left( u_i - \log \left( \frac{w_i f_i - w_i}{z_i} \right) + \frac{|w_i|^2}{2\sigma^2} + \alpha \| \nabla u_i \| \right),
\]

where the constant terms which do not affect the minimisation over \(u\) and \(w\) have been neglected. Passing from a discrete to a continuous modelling similarly as described in the discussion for the additive case, we obtain the following continuous model

\[
\min_{u,w} |D u|_2(\Omega) + \frac{\lambda_1}{2} \|w\|_{L^2(\Omega)}^2 + \lambda_2 \int_{\Omega} \left( u(x) - \log \left( \frac{u(x)^{f(x)-w(x)}}{(f(x) - w(x))^!} \right) \right) \, d\mu(x), \tag{3.14}
\]

where we have set \(\lambda_1 := 1/\alpha\sigma^2\) and \(\lambda_2 := 1/\alpha\) and we have still to specify the function spaces where the minimisation takes place. Using the standard Stirling approximation of the logarithm of the factorial function, we rewrite the second term in (3.14) as

\[
\int_{\Omega} \left( u(x) - \log \left( \frac{u(x)^{f(x)-w(x)}}{(f(x) - w(x))^!} \right) \right) \, d\mu(x)
\]

\[
= \int_{\Omega} \left( u(x) - (f(x) - w(x)) \log (u(x)) + \log \left( \left( f(x) - w(x) \right)^! \right) \right) \, d\mu(x)
\]

\[
\approx \int_{\Omega} \left( u(x) - (f(x) - w(x)) \log \left( \frac{f(x) - w(x)}{u(x)} \right) - (f(x) - w(x)) \right) \, d\mu(x) = D_{KL}(f - w, u),
\]

i.e. the Kullback-Leibler (KL) functional (2.3). We refer the reader to Section 2 of the supplementary material where some properties of \(D_{KL}\) are recalled. The variational Poisson-Gaussian data fidelity (3.14) can then be written in a more compact form as

\[
\min_{u \in B V(\Omega), \ A} \min_{w \in L^2(\Omega) \cap B} |D u|_2(\Omega) + \frac{\lambda_1}{2} \|w\|_{L^2(\Omega)}^2 + \lambda_2 D_{KL}(f - w, u), \tag{3.15}
\]

where the admissible sets \(A\) and \(B\) are chosen as specified in (2.5) to guarantee that all the terms in (3.15) are well-defined.

In both the general model consistently with our statistical assumptions, model the single noise components by the fidelity terms are weighted against each other by the parameters \(\lambda_1\) and \(\lambda_2\), whose size depends on the intensity of each noise component, and weight the fidelity against the TV regularisation.

In the following, we discuss a different interpretation of the TV-IC model, as the MAP estimate for a posterior distribution in which the two noise distributions are combined by the so-called infinity convolution operation, a modification of the expression of the classical convolution one describing the probability density of the sum of two random variables.

### 3.2 Infinity convolution

Given two independent real-valued random variables \(V\) and \(W\) with associated probability densities \(f_V\) and \(f_W\), it is well-known that the random variable \(Z := V + W\) has probability density \(f_Z\) given by the convolution of \(f_V\) and \(f_W\), that is

\[
f_Z(z) = \int_{\mathbb{R}} f_V(v) f_W(z - v) \, dv = (f_V * f_W)(z). \tag{3.16}
\]
Following [31, Remark 2.3.2], since $f_V$ and $f_W$ are non-negative, we can define for $p > 0$ the convolution of order $p$ as follows:

$$f_Z^p(z) := (f_V *_p f_W)(z) = \left( \int_{\mathbb{R}} (f_V(v)f_W(z - v))^p \, dv \right)^{1/p}. \quad (3.17)$$

This clearly corresponds to the classical convolution \( (3.16) \) if $p = 1$. By now letting $p \to \infty$ it is easy to show that \( (3.17) \) converges to the infinity convolution of $f_V$ and $f_W$

$$f_Z^\infty(z) := \sup_{v \in \mathbb{R}} f_V(v)f_W(z - v). \quad (3.18)$$

We note that $f_Z(z)$ in \( (3.16) \) and $f^\infty$ have the same domain. Also, note that in order for $f^\infty$ to be a probability density the following normalisation

$$\tilde{f}_Z^\infty(z) := \frac{1}{\int f_Z^\infty(z) \, dz} f_Z^\infty(z)$$

is needed. Then, $\tilde{f}_Z^\infty$ is a probability density, having the same domain of $f_Z$, but potentially featuring heavier tails.

In the special case when the probability densities $f_V$ and $f_W$ have an exponential form of the type $f_V(\cdot) = c_V e^{-g_V(\cdot)}$ and $f_W(\cdot) = c_W e^{-g_W(\cdot)}$, with $c_V$ and $c_W$ being positive constants and $g_V, g_W : \mathbb{R} \to \mathbb{R}^+$ continuous and convex positive functions, \( (3.18) \) can be equivalently rewritten as:

$$f_V *_\infty f_W(z) = c_V c_W e^{-\inf_{v \in \mathbb{R}} (g_V(v) + g_W(z-v))}. \quad (3.19)$$

Hence, the infinity convolution of $f_V$ and $f_W$ corresponds to the infimal convolution of $g_V$ and $g_W$ through a negative exponentiation and up to multiplication by positive constants.

Let us now recall the additive model \( (3.8) \). At every pixel, $i = 1, \ldots, M$ the random variable $Y_i := V_i + W_i$ is the sum of the two independent random variables $V_i$ and $W_i$ having Laplace and Gaussian probability densities \( (3.9) \), respectively. We want to compute the MAP estimate for the modified infinity convolution likelihood \( (3.19) \), optimising over $u$ only. By independence of the realisations, we have that the probability density $f_Y$ of the random vector $Y = (Y_1, \ldots, Y_M)$ in correspondence with the realisation $y = (y_1, \ldots, y_M)$ reads

$$f_Y(y) = \prod_{i=1}^M f_Y(y_i). \quad (3.20)$$

Each probability density $f_Y(y_i)$ is formally given by the convolution between the Laplace and Gaussian probability distribution. However, if we replace this convolution with the infinity convolution defined in \( (3.18) \) and normalise appropriately in order to get a probability distribution, we get

$$f_Y(y_i) = \frac{1}{2\tau \sqrt{2\pi} \sigma^2} e^{-\inf_{v_i \in \mathbb{R}} (|v_i|/\tau + |y_i - v_i|^2/2\sigma^2)} \quad \tilde{f}_Y(y_i) := \frac{f_Y(y_i)}{\sum_{i=1}^M f_Y(y_i)}$$

for every $i = 1, \ldots, M$. Plugging this expression in \( (3.20) \) and computing the negative log-likelihood of $P(y|u)$, we get:

$$-\log P(y|u) = -\log \prod_{i=1}^M \tilde{f}_Y(y_i) = -\sum_{i=1}^M \log(\tilde{f}_Y(y_i)) = -\sum_{i=1}^M \log(\tilde{f}_Y(f_i - u_i)).$$

Thus, we have that the log-likelihood we intend to minimise is

$$\sum_{i=1}^M \inf_{v_i \in \mathbb{R}} \left( \frac{|v_i|}{\tau} + \frac{|f_i - u_i - v_i|^2}{2\sigma^2} \right),$$

where the constant terms which do not affect the minimisation over $u$ have been neglected.
Similarly as before, passing from a discrete to a continuous representation we get

$$\int_\Omega \inf_{v(x) \in \mathbb{R}} \left( \frac{|v(x)|}{\tau} + \frac{|f(x) - u(x) - v(x)|^2}{2\sigma^2} \right) \, d\mu(x)$$

$$= \inf_{v \in L^2(\Omega)} \int_\Omega \left( \frac{|v(x)|}{\tau} + \frac{|f(x) - u(x) - v(x)|^2}{2\sigma^2} \right) \, d\mu(x),$$

(3.21)

where the infimum and the integral operators commute by assuming $v$ is $L^2(\Omega)$ (see [28] for further details). This also ensures that the integrand terms are both well defined. Defining $\lambda_1 := 1/\tau$ and $\lambda_2 := 1/\sigma^2$ we derive from (3.21) the same data fidelity term as in (3.11).

The same computation can be done for the signal dependent case of the Poisson-Gaussian noise mixture [312] with [313], thus obtaining the same model (3.15).

### 3.3 Connection with existing approaches

Several variational models for image denoising in the presence of combined noise distributions have been considered in the literature.

In the case of a mixture of impulsive and Gaussian noise, these models can be roughly divided in two categories. The first category considers an additive $L^1 + L^2$ data modelling. In [30], for instance, the image domain is decomposed in two parts, with impulsive noise in one and Gaussian noise in the other, and the data discrepancy is modelled by the sum of an $L^1$ and $L^2$ data fidelity supported on the respective domain with Gaussian or impulse noise. For the numerical solution an efficient domain decomposition approach is used. Similarly, in [22, 13, 12] semi-smooth Newton’s methods are employed to solve a denoising model where $L^1$ and $L^2$ data fidelities are combined in an additive fashion. A second category of methods renders the removal of impulsive and Gaussian noise in a two-step procedure (see, e.g. [11]). In the first phase the pixels damaged by the impulsive noise component are identified via an outlier-removal procedure and removed by means of a $\text{TV}-L^1$ model. Then a second denoising step with an $L^2$ data fidelity is employed for removing the Gaussian noise in all other pixels. These methods are often presented as image inpainting strategies where the pixels corrupted by impulsive noise are first identified and then filled in using the information nearby with a variational regularisation approach. A similar approach is considered in [23, 51] where the impulse noise removal step using framelet transformation of the noisy image. In these works, the structure of the resulting data discrepancy is similar to (2.2), although the problem is posed in a finite dimensional setting with no statistical motivation for the particular choice of the mixed data discrepancy considered.

When a mixture of Gaussian and Poisson noise is assumed, an exact log-likelihood model has been considered in [33, 34]. In the same discrete setting described above, still denoting by $M$ the total number of pixels, the expression of the negative log-likelihood proposed reads

$$\Phi_{PG}(u,f) = \sum_{i=1}^M \left( -\log \sum_{\pi=0}^{\infty} \frac{u_i^\pi e^{-u_i}}{\pi!} e^{-\left( \frac{L_i}{\sqrt{\pi\sigma}} \right)^2} \right).$$

(3.22)

In comparison with our derivation presented in Section 3.2, here the log-likelihood $\Phi_{PG}(u,f)$ is exact. As discussed in Section 3, the model we propose is derived either by a joint MAP estimation or by replacing the convolution with the infinity convolution. The resulting variational model (3.15) is much simpler than (3.22) and is amenable for an efficient numerical realisation. In [33, 34] the authors solve the model efficiently by first splitting $\Phi_{PG}(u,f)$ into the sum of two different terms, the former being a convex Lipschitz-differentiable function, the latter being a proper, convex and lower semi-continuous function. Then a first-order optimisation method based on the use of primal-dual splitting algorithms is design. However, due to the approximation of the infinite sum in (3.22), truncation errors need to be controlled and computational efficiency may suffer.

Another approach for mixed Gaussian and Poisson noise is considered in [37], where the authors design a data fidelity term similar to (3.15) combining it with TV regularisation for image denoising. Despite the analogies between the joint MAP estimation of their and our model, in their work no well-posedness results in function spaces nor properties of noise decompositions are discussed. Nonetheless,
in [37] the good performance of the combined model is observed in terms of improvements of the Peak Signal to Noise Ratio (PSNR) for several synthetic and microscopy images. The simple structure of our models (3.11) and (3.15) allows for the design of efficient first and second-order numerical schemes. In this paper we will focus on the latter (see Section 6). Our approach is further able to ‘decompose’ the noise into its different statistical components, each one corresponding to one particular noise distribution in the data.

4 Well-posedness of the TV-IC model

Thanks to Proposition 2.2 and Proposition 2.4 we can conclude that in both cases (2.2) and (2.4), the function \( \Phi^{\lambda_1,\lambda_2} \) is well-defined, i.e. for every \( u \in BV(\Omega) \cap A \) there exists a unique element \( v^*(u) \in L^2(\Omega) \cap B \) minimising the functional \( \mathcal{F}^{\lambda_1,\lambda_2}(f, u, \cdot) \). Problem (TV-ICa) can then be rewritten as

\[
\min_{u \in BV(\Omega) \cap A} \{ J(u) := |Du|_{L^1}(\Omega) + \lambda_1 \Phi_1(v^*(u)) + \lambda_2 \Phi_2(u, f - v^*(u)) \},
\]

(4.1)

where \( v^*(u) \in L^2(\Omega) \cap B \) is the unique solution of (TV-ICb) in the two cases (2.2) and (2.4). In particular, for every \( u \in BV(\Omega) \cap A \), there is a positive finite constant \( C = C(u) \) such that

\[
\|v^*(u)\|_{L^2(\Omega)} \leq C(u).
\]

(4.2)

Note that the constant in (4.2) may depend on \( u \) and hence does not necessarily bound \( v \) uniformly. Therefore, for the following existence proof, we restrict the admissible set of solutions for \( v \) by intersecting it with the closed ball in \( L^2(\Omega) \) centre in \( f \) and with fixed radius \( R > 0 \). That is, in (TV-ICb) we define the new admissible set

\[
\tilde{B} := L^2(\Omega) \cap (\mathcal{B} \cap B_R(f)),
\]

(4.3)

where \( B_R(f) := \{ z \in L^2(\Omega) : \|z - f\|_{L^2(\Omega)} \leq R \} \) for some \( R > 0 \) and \( \mathcal{B} \) is defined as before in each case. Since \( B_R(f) \) is compact and convex, the well-posedness properties studied in Section 2 still hold true. In addition, one can now easily compute the constant \( C \) of (4.2) by Young’s inequality. Namely, for every \( u \in BV(\Omega) \cap A \) one has now

\[
\|v^*(u)\|_{L^2(\Omega)} \leq C,
\]

(4.4)

with \( C := \sqrt{2(R^2 + \|f\|^2_{L^2(\Omega)})} \). This additional assumption is reasonable for our applications since we want the noise component \( v \) to preserve some similarities with the given image \( f \) in terms of its noise features.

After this modification, we can now state and prove the main existence result for both noise combinations described in Section 2.1 and 2.2 by means of standard tools of calculus of variations. We refer the reader also to [50] and [8] where similar results are proved for \( L^1 \) and Kullback-Leibler-type data fidelities, respectively.

Theorem 4.1. Let \( \lambda_1, \lambda_2 > 0 \) and let us denote by \( v^*(u) \in \tilde{B} \) defined in (4.3) the solution of the minimisation problem (TV-ICb) in one of the two cases (2.2) and (2.4) provided by Propositions 2.2 and 2.4 respectively, for every \( u \in BV(\Omega) \cap A \). Then, the minimisation problem (4.1) has a minimiser.

Proof. Let \( \{u_n\} \subset BV(\Omega) \cap A \) be a minimising sequence for \( J \). Such sequence exists since the functional \( J \) is non-negative. Neglecting the positive contribution given by \( \Phi_1 \), we have:

\[
|Du_n|_{L^1}(\Omega) + \lambda_2 \Phi_2(u_n, f - v^*(u_n)) \leq |Du_n|_{L^1}(\Omega) + \Phi^{\lambda_1,\lambda_2}(u_n, f) \leq M, \quad \text{for all } n
\]

(4.5)

for some finite constant \( M > 0 \). To show the uniform boundedness of the sequence \( \{u_n\} \) in \( BV(\Omega) \) we first observe that using the positivity of \( \Phi_2 \), (4.5) we have

\[
|Du_n|_{L^1}(\Omega) \leq M, \quad \text{for all } n.
\]

(4.6)

Next, in order to get appropriate bounds for \( \{u_n\} \) in \( L^1(\Omega) \) we need to differentiate the two cases considered.
For $\Phi_2(u_n, f - v^*(u_n)) = \frac{1}{2} \|f - v^*(u_n) - u_n\|_{L^2(\Omega)}^2$, we get from (4.5), and by Young’s inequality:

$$M \geq \Phi_2(u_n, f - v^*(u_n)) = \frac{1}{2} \|f - u_n - v^*(u_n)\|_{L^2(\Omega)}^2 \geq C_1 \|u_n\|_{L^1(\Omega)}^2 - C_2,$$

where the constants $C_1 := \frac{1}{4|\Omega|}$ and $C_2 := \frac{1}{2} R^2$ are finite by (4.4). Hence, $\{u_n\}$ is bounded in $L^1(\Omega)$, which, combined with (4.6) gives us that the sequence $\{u_n\}$ is bounded in $BV(\Omega)$.

- **Gaussian-salt & pepper case:** For $\Phi_2(u_n, f - v^*(u_n)) = \frac{1}{2} \|f - v^*(u_n) - u_n\|_{L^2(\Omega)}^2$, we get from (4.5) and by Young’s inequality:

$$M \geq \Phi_2(u_n, f - v^*(u_n)) = \frac{1}{2} \|f - u_n - v^*(u_n)\|_{L^2(\Omega)}^2 \geq C_1 \|u_n\|_{L^1(\Omega)}^2 - C_2,$$

where the constants $C_1 := \frac{1}{4|\Omega|}$ and $C_2 := \frac{1}{2} R^2$ are finite by (4.4). Hence, $\{u_n\}$ is bounded in $L^1(\Omega)$, which, combined with (4.6) gives us that the sequence $\{u_n\}$ is bounded in $BV(\Omega)$.

- **Gaussian-Poisson case:** Let $g_n := f - v^*(u_n)$. Under the choice $\Phi_2(u_n, g_n) = D_{KL}(g_n, u_n)$, and using the uniform bound (4.3) on $g_n$ we can apply directly the $BV$-coercivity result for the standard TV-KL functional proved in [10] Lemma 6.3.2 and get from (4.5):

$$C \|u_n\|_{BV(\Omega)} \leq \|Du_n\|_{(\Omega)} + \lambda_2 \ D_{KL}(g_n, u_n) \leq J(u_n) \leq M,$$

for every $n$ to conclude that $\{u_n\}$ is bounded in $BV(\Omega)$.

Thanks to the uniform boundedness of the sequence $\{u_n\}$ in $BV(\Omega) \subset L^1(\Omega)$, we have that, up to a non-relabelled subsequence, there is $u \in BV(\Omega)$ such that:

$$u_n \rightharpoonup u \text{ in } BV(\Omega), \quad u_n \rightarrow u \text{ in } L^1(\Omega).$$

Since the image domain $\Omega$ is bounded, a further non-relabelled subsequence $\{u_n\}$ converging point wise to $u$ a.e. in $\Omega$ can be extracted. We claim now that $u$ is a non-relabelled subsequence, there is $\hat{u} \in BV(\Omega)$ of $\{u_n\}$ such that:

$$u_n \rightharpoonup \hat{u} \text{ in } BV(\Omega), \quad u_n \rightarrow \hat{u} \text{ in } L^1(\Omega).$$

- **Gaussian-salt & pepper case:** Recalling Remark 2.1 and [3] Proposition 12.15, we have that the data fidelity $\Phi^{\lambda_1, \lambda_2}(u, f)$ is continuous by standard properties of Moreau envelope.

- **Gaussian-Poisson case:** Recalling Proposition 2.3 in Section 2 of the supplementary material, we observe that the functional $\Phi_2(u_n, f - v^*(u_n)) = D_{KL}(f - v^*(u_n), u_n)$ is weakly lower semi-continuous in $L^1(\Omega)$ in both arguments. In fact, by assumptions (2.5) and (4.3) on the admissible sets $A$ and $B$, we have that for every $n$ the first argument $f - v_n$ is non-negative and integrable and the second argument $u_n$ is non-negative and bounded in $L^1$ from what is shown above. Let us now define for every element $u_n \in BV(\Omega) \cap A$ the corresponding unique solution $v_n$ of the minimisation problem (2.4) provided by Proposition 2.4, so let $v_n := v^*(u_n)$ for every $n$. By the uniform estimate (4.4), we have

$$\|v_n\|_{L^2(\Omega)} \leq C \quad \text{for every } n.$$
Remark 4.2. In the case of the $L^1$-$L^2$ TV-IC model (2.2), we can alternatively show well-posedness using a similar argument as in [10, Lemma 2.4, Proposition 3.2] by considering the IC 1-homogeneous functional $\Phi_{1-hom}^{\lambda_1,\lambda_2}$ defined as:

$$\Phi_{1-hom}^{\lambda_1,\lambda_2}(u, f) := \min_{v \in L^2(\Omega)} \lambda_1 \|v\|_{L^1(\Omega)} + \lambda_2 \|f - v\|_{L^2(\Omega)}.$$ 

By standard lower semi-continuity arguments, one can show that the minimum above is attained. By considering the analogous TV-type problem:

$$\min_{u \in BV(\Omega) \cap A} \left\{ |Du|(\Omega) + \Phi_{1-hom}^{\lambda_1,\lambda_2}(u, f) \right\},$$

one can further show that for every minimising sequence $\{u_n\} \subset BV(\Omega) \cap A$, an estimate similar to (4.5) implies that for every $n$ and every $w \in L^2(\Omega)$:

$$\|u_n\|_{L^1(\Omega)} \leq \|u_n - f\|_{L^1(\Omega)} + \|w\|_{L^1(\Omega)}$$

$$\leq \|u_n - f - w\|_{L^1(\Omega)} + |\Omega|^{1/2} \|w\|_{L^2(\Omega)} \leq C_1 \left( \|u_n - f - w\|_{L^1(\Omega)} + \|w\|_{L^2(\Omega)} \right).$$

where $C_1 := \max \{ |\Omega|^{1/2}, 1 \}$. Hence, for an appropriate choice of $C_2$ we can get

$$C_2 \|u_n\|_{L^1(\Omega)} \leq \lambda_1 \|u_n - f\|_{L^1(\Omega)} + \lambda_2 \|w\|_{L^2(\Omega)} \leq M + C_2 \|f\|_{L^1(\Omega)} < \infty \quad \text{for every } n,$$

for every $w \in L^2(\Omega)$ and for $M > 0$. In this case, we then do not need to restrict to the set $\hat{\mathcal{B}}$ as in (4.3), since the bound on $\{u_n\}$ in $L^1(\Omega)$ is uniform by standard $L^p$ inclusions. To conclude, one can then apply a similar result as [10, Proposition 3.2] which ensures that the 1-homogeneous problem (4.7) and its 2-homogeneous version with data fidelity (2.2) have in fact the same minimisers. We remark here that since the same argument does not apply to the case of Kullback-Leibler functional, the restriction to the set $\hat{\mathcal{B}}$ (4.3) is needed to prove the theorem 4.1 for the TV-IC $L^2$-KL case (2.4).

### 4.1 Well-posedness of Huber-regularised TV-IC

In view of the numerical realisation of the TV-IC via a gradient-based method presented in Section [6] we smooth the TV energy to avoid the multivaluedness of its subdifferential. In particular, we consider a standard Huber-type smoothing of TV depending on a parameter $\gamma \gg 1$. For a general function $z : \Omega \rightarrow \mathbb{R}^\ell$, the Huber regularisation of $|z|$ is defined as:

$$|z|_\gamma := \begin{cases} |z| - \frac{1}{2\gamma}, & \text{if } |z| \geq \frac{1}{\gamma} \\ \frac{\gamma}{2} |z|^2, & \text{if } |z| < \frac{1}{\gamma}. \end{cases}$$

(4.8)

In words, in the proximity of the points where $|z|$ is small a quadratic smoothing is used, while essentially the function is kept the same for large values of $|z|$. Clearly, the non-smooth version is recovered letting $\gamma \rightarrow \infty$. Denoting by $dx$ the usual Lebesgue measure in $\mathbb{R}^2$ and by $\mathcal{B}(\Omega)$ the $\sigma$-algebra of $\Omega$, let $Du = \nabla u \, dx + D_\omega u$ be the Lebesgue decomposition of the two-dimensional distributional gradient $Du$ into its absolutely continuous $\nabla u \, dx$ and singular $D_\omega u$ parts. Then, the Huber-regularisation of the total variation measure $|Du|$ is defined as

$$|Du|_\gamma(V) := \int_V |\nabla u|_\gamma \, dx + \int_V |D_\omega u|, \quad V \in \mathcal{B}(\Omega),$$

(4.9)

i.e. the absolutely continuous part is regularised using (4.8). The Huber-regularised version of (4.1) reads:

$$\min_{u \in BV(\Omega) \cap A} \left\{ J_\gamma(u) := |Du|_\gamma(\Omega) + \lambda_1 \Phi_1(v^*(u)) + \lambda_2 \Phi_2(u, f - v^*(u)) \right\},$$

(4.10)

where, as before, for every $u \in L^2(\Omega) \cap A$ the element $v^*(u) \in \hat{\mathcal{B}}$ is the unique solution of (TV-ICb) and $\hat{\mathcal{B}}$ is defined as in (4.3). As a Corollary of Theorem 4.1 we show now that also the regularised TV-IC problem is well-posed.
Corollary 4.3. Let $\lambda_1, \lambda_2 > 0$ and let $v^*(u) \in \hat{B}$ be the solution of the minimisation problem (TV-ICb) for every $u \in BV(\Omega) \cap A$. Then, the Huber-regularised minimisation problem (4.10) has a minimiser $u \in BV(\Omega) \cap A$.

Proof. The Huber regularisation function is coercive and has at most linear growth (cf. [22, Theorem 2.1]). Forgetting the contribution coming from the positive term $\Phi_2$, we have:

$$|Du_n|(\Omega) + \lambda_2 \Phi_2(u_n, f - v^*(u_n))$$

$$\leq |Du_n|_{\gamma}(\Omega) + \lambda_1 \Phi_1(v^*(u_n)) + \lambda_2 \Phi_2(u_n, f - v^*(u_n)) \leq M,$$

for every minimising sequence $\{u_n\}$ in $BV(\Omega) \cap A$. To conclude, we can simply use the result proved in Theorem 4.1 for the non-smooth case combined with the lower-semicontinuity property of $|Du|_{\gamma}(\Omega)$ with respect to the strong topology of $L^1(\Omega)$ (see, e.g., [21]).

We now connect the solution of the regularised minimisation problem (4.10) to a solution of the non-smooth problem (4.1) via $\Gamma$-convergence [19].

Theorem 4.4. The sequence of functionals $J_{\gamma} : BV(\Omega) \times A \to \mathbb{R}$ defined in (4.10) $\Gamma$-converges to the functional $J : BV(\Omega) \times A \to \mathbb{R}$ defined in (4.1) as $\gamma \to \infty$. Hence, if $\{u_{\gamma_n}\}_{n \in \mathbb{N}} \subset BV(\Omega) \cap A$ is a sequence of minimisers of $J_{\gamma}$, then every subsequence of $\{u_{\gamma_n}\}_{n \in \mathbb{N}}$ has a weak* -cluster point in $BV(\Omega) \cap A$ which is a minimiser of $J$ as $\gamma \to \infty$.

Proof. The proof is a standard result based on relaxation techniques for measures, see e.g. [6]. We observe that as $\gamma \to \infty$ the functional $J_{\gamma}$ converges for every $u \in BV(\Omega) \cap A$, to

$$J(u) = \int_{\Omega} |\nabla u| \, dx + \int_{\Omega} |Du|_\gamma \, dx + \int_{\Omega} |Du|_\gamma \, dx + \lambda_1 \Phi_1(v^*(u)) + \lambda_2 \Phi_2(u, f - v^*(u))$$

since for the absolutely continuous part $\nabla u$ there holds $|\nabla u|_{\gamma} \to |\nabla u|$ pointwise as $\gamma \to \infty$. Since the convergence is monotonically increasing with respect to $\gamma$ (see Figure 1), we have that $J_{\gamma}$ $\Gamma$-converges to $J$ (see [19, Remark 5.5]). We can then conclude that any subsequence extracted from $\{u_{\gamma_n}\}_{n \in \mathbb{N}}$ has a weak* -cluster point in $BV(\Omega) \cap A$ which must be a minimiser of $J$ by [19, Corollary 7.20].

In the following, we will focus for simplicity on the equivalent formulation of the nonsmooth problem (4.1) given by:

$$\min_{u \in BV(\Omega) \cap A} \left\{ J(u, v) := |Du|_{\gamma}(\Omega) + \lambda_1 \Phi_1(v) + \lambda_2 \Phi_2(u, f - v) \right\},$$

where the two components of the model, i.e. the reconstructed image $u$ and the noise component $v$ are treated jointly. Analogously, in Section 6 we will consider the corresponding Huber-regularised version

$$\min_{u \in BV(\Omega) \cap A} \left\{ J_{\gamma}(u, v) := |Du|_{\gamma}(\Omega) + \lambda_1 \Phi_1(v) + \lambda_2 \Phi_2(u, f - v) \right\},$$

and use it for the design of efficient gradient-based numerical schemes.
5 Recovery of the single noise models

In this section we show asymptotic results for solutions $(u^*, v^*)$ of (4.11) as the parameters $\lambda_1$ and $\lambda_2$ become infinitely large. We discuss the Gaussian-salt & pepper (see Section 2.1) and the Gaussian-Poisson case (see Section 2.2) separately for more clarity. A finer study on the structure of solutions of the TV-IC model similar to the ones in [10] will be the topic of a forthcoming paper [14].

5.1 The Gaussian-salt & pepper case

The following proposition asserts essentially that TV-$L^2$ [44] and TV-$L^1$ [10] [26]-type models can be recovered ‘asymptotically’ from (4.11) by letting the $L^1/L^2$ fidelity weight become infinitely large, respectively. Moreover, when both parameters become infinitely large, a full recovery of the data is obtained.

**Proposition 5.1.** Let $(u^*, v^*) \in BV(\Omega) \times L^2(\Omega)$ be an optimal pair for (4.11) in the Gaussian-salt & pepper case described in Section 2.1. If $f \in L^2(\Omega)$ is not identically zero, then the following asymptotic convergences hold:

i) If $\lambda_2$ is finite, then $v^* \to 0$ in $L^1(\Omega)$ as $\lambda_1 \to +\infty$.

ii) If $\lambda_1$ is finite, then $v^* \to f - u^*$ in $L^2(\Omega)$ as $\lambda_2 \to +\infty$.

iii) If, additionally, $f \in BV(\Omega)$ and is not a constant, then the pair $(u^*, v^*)$ converges to $(f, 0)$ in $L^1(\Omega) \times L^1(\Omega)$ as $\lambda_1, \lambda_2 \to +\infty$.

**Proof.** In correspondence of an optimal pair $(u^*, v^*) \in BV(\Omega) \times L^2(\Omega)$ the following variational inequality holds:

$$|Du^*|(\Omega) + \lambda_1 \|v^*\|_{L^1(\Omega)} + \frac{\lambda_2}{2} \|f - u^* - v^*\|_{L^2(\Omega)}^2 \leq$$

$$|Du|(\Omega) + \lambda_1 \|v\|_{L^1(\Omega)} + \frac{\lambda_2}{2} \|f - u - v\|_{L^2(\Omega)}^2,$$

for all $u \in BV(\Omega)$ and $v \in L^2(\Omega)$. (5.1)

For the proof of i) we choose in (5.1) $u = v = 0$. We have:

$$\lambda_1 \|v^*\|_{L^1(\Omega)} \leq |Du^*|(\Omega) + \lambda_1 \|v^*\|_{L^1(\Omega)} + \frac{\lambda_2}{2} \|f - u^* - v^*\|_{L^2(\Omega)}^2 \leq \frac{\lambda_2}{2} \|f\|_{L^2(\Omega)}^2,$$

which implies:

$$\|v^*\|_{L^1(\Omega)} \leq \frac{\lambda_2}{2 \lambda_1} \|f\|_{L^2(\Omega)}^2.$$

Since $\lambda_2$ is positive and finite, by letting $\lambda_2$ go to infinity, we have that $v^*$ converges to 0 in $L^1(\Omega)$.

Similarly, for ii) we choose $u = 0$ and $v = f$ in (5.1) and get:

$$\|f - u^* - v^*\|_{L^2(\Omega)}^2 \leq \frac{2 \lambda_1}{\lambda_2} \|f\|_{L^1(\Omega)}^2.$$

The conclusion follows analogously by letting $\lambda_2$ go to infinity.

For the proof of iii) we choose in (5.1) $u = f \in BV(\Omega)$ and $v = 0$. Since $f$ is not a constant, $|Df|(\Omega) \neq 0$ and

$$C_{\lambda_1, \lambda_2} \left( \|v^*\|_{L^1(\Omega)} + \|f - u^* - v^*\|_{L^2(\Omega)}^2 \right) \leq |Du^*|(\Omega) + \lambda_1 \|v^*\|_{L^1(\Omega)} + \frac{\lambda_2}{2} \|f - u^* - v^*\|_{L^2(\Omega)}^2 \leq \frac{\lambda_2}{2} \|Df\|(\Omega),$$

where $C_{\lambda_1, \lambda_2} := \min \left\{ \lambda_1, \frac{\lambda_2}{2} \right\}$. We have:

$$\|v^*\|_{L^1(\Omega)} + \|f - u^* - v^*\|_{L^2(\Omega)}^2 \leq \frac{1}{C_{\lambda_1, \lambda_2}} |Df|(\Omega).$$

The right hand side of the inequality above goes to zero as $\lambda_1, \lambda_2$ tend to $+\infty$ since in this case $C_{\lambda_1, \lambda_2}$ goes to infinity. Thus, as $\lambda_1, \lambda_2 \to +\infty$ we have the following convergences:

$$v^* \to f - u^* \text{ in } L^2(\Omega), \quad v^* \to 0 \text{ in } L^1(\Omega).$$

By uniqueness of the limit in $L^1(\Omega)$ we conclude that $f - u^* = 0$ and.

\[ \square \]
5.2 The Gaussian-Poisson case

In the Gaussian-Poisson framework described in Section 2.2 similar results can be proved. They rely on analogous energy estimates and, essentially, on an estimate for the KL fidelity term (2.3) recalled in Equation (2.2) of Section 2 in the supplementary material.

Proposition 5.2. Let $\mathcal{A}$ and $\mathcal{B}$ be the admissible sets in (2.5). Let $(u^*, v^*) \in (BV(\Omega) \cap \mathcal{A}) \times (L^2(\Omega) \cap \mathcal{B})$ one optimal pair for (4.11) in the Gaussian-Poisson case described in Section 2.2. If $f \in L^\infty(\Omega)$ is not identically zero, then the following asymptotic convergences hold:

i) If $\lambda_2$ is finite and $f$ is not identically equal to one, then $v^* \rightarrow 0$ in $L^2(\Omega)$ as $\lambda_1 \rightarrow +\infty$.

ii) If $\lambda_1$ is finite, then $v^* \rightarrow f - u^*$ in $L^1(\Omega)$ as $\lambda_2 \rightarrow +\infty$.

iii) If additionally $f \in BV(\Omega)$ and is not a constant, then the pair $(u^*, v^*)$ converges to $(f, 0)$ in $L^1(\Omega) \times L^2(\Omega)$ as $\lambda_1, \lambda_2 \rightarrow +\infty$.

Proof. Again, we start writing explicitly the variational inequality satisfied by every optimal pair $(u^*, v^*) \in (BV(\Omega) \cap \mathcal{A}) \times (L^2(\Omega) \cap \mathcal{B})$ for the Gaussian-Poisson combined case:

\[
|Du^*|(\Omega) + \frac{\lambda_1}{2} \|v^*\|^2_{L^2(\Omega)} + \lambda_2 \, D_{KL}(f - v^*, u^*) \\
\leq |Du|(\Omega) + \frac{\lambda_1}{2} \|v\|^2_{L^2(\Omega)} + \lambda_2 \, D_{KL}(f - v, u), 
\]

for all $u \in (BV(\Omega) \cap \mathcal{A})$ and $v \in (L^2(\Omega) \cap \mathcal{B})$.

For the proof of i) we choose in (5.2) $u = 1_\Omega$, the constant function identically equal to one on $\Omega$ and $v = 0$. We deduce:

\[
\frac{\lambda_1}{2} \|v^*\|^2_{L^2(\Omega)} \leq \lambda_2 D_{KL}(f, 1_\Omega).
\]

Since by assumption the function $f$ is not identically equal to one, the right hand side of the inequality above is strictly positive and bounded as:

\[
0 < D_{KL}(f, 1_\Omega) \leq K := \left(\|f\|_{L^\infty(\Omega)} \|\log f\|_{L^1(\Omega)} + |\Omega|\right) < \infty,
\]

by Hölder inequality and assumptions on $f$. Thus, we have:

\[
\|v^*\|^2_{L^2(\Omega)} \leq \frac{2\lambda_2}{\lambda_1} K,
\]

which implies the convergence $v^* \rightarrow 0$ in $L^2(\Omega)$ as $\lambda_1 \rightarrow +\infty$.

To prove ii), we consider in (5.2) $u = 0$ and $v = f$. For such a choice we have $D_{KL} = 0$. Hence, inequality (5.2) reduces to:

\[
\lambda_2 \, D_{KL}(f - v^*, u^*) \leq \frac{\lambda_1}{2} \|f\|^2_{L^2(\Omega)},
\]

which implies that $D_{KL}(f - v^*, u^*) \rightarrow 0$ as $\lambda_2 \rightarrow +\infty$. Thanks to Corollary (2.4) in the supplementary material, we deduce that $v^* \rightarrow f - u^*$ in $L^1(\Omega)$.

We can show that iii) holds by choosing $v = 0$ and $u = f \in BV(\Omega)$ with $|Df|(\Omega) \neq 0$ by assumption. Proceeding similarly as before, the convergence result follows immediately after applying once again Corollary (2.4) in the supplementary material. \[\square\]

6 Numerical results

In this section we report on the numerical realisation of the mixed noise model (TV-ICA)-(TV-ICB) in the form (4.11) for the two frameworks described in Sections 2.1 and 2.2.

We consider a discretised image domain $\Omega = \{(x_k, y_l) : k = 1, \ldots, K, \ l = 1, \ldots, L\}$ with cardinality $|\Omega| = K L$. Standard finite difference discretisation schemes are used. In particular, forward and backward finite differences are considered for the discretisation of the divergence and gradient operators,
respectively, thus preserving their mutual adjointness property (similarly as in [15], for instance). For the numerical realisation of the model, we consider the optimality system of the Huber-regularised joint formulation (4.12) and solve it numerically using a SemiSmooth Newton (SSN) type algorithm with a primal-dual strategy. Relations with the original problem are guaranteed by Theorem 4.4. We refer the reader to [21] for more details on Newton-type methods in PDE-constrained optimisation and to [22, 12] for similar approaches in an imaging framework. First-order convex optimisation methods [17] could alternatively be used. In this paper we chose a second-order scheme in view of a parameter learning estimation via bilevel optimisation as outlined in Section 7.1.

**Test images and parameters** For our computational tests we consider different images selected either from the Berkeley database\(^1\), see Figure 2, or from some other public available website, see Figure 3. For each experiment, the ground truth image \(u_0\) is artificially corrupted with mixed noise distributions of different intensities which are specified in each case. For simplicity, we consider square \(K \times K\) pixel images (corresponding to a step size \(h = 1/K\)) and fix the Huber-regularisation parameter to be \(\gamma = 1e5\). In our algorithms we use a combined stopping criterion which stops the iterations either when the norm of the difference between two different iterates is below a given tolerance \(\epsilon = 1e-7\) or when a maximum number of iterations (typically, 35) is attained.

![Figure 2: Some images from the Berkeley database.](image1)

![Figure 3: Some additional images used for our experiments\(^2\).](image2)

### 6.1 Gaussian-salt & pepper case

We start focusing on the Gaussian-salt & pepper model considered in Section 2.1. The Huber-regularised minimisation problem reads

\[
\min_{u,v} \left\{ J_\gamma(u, v) := \|\nabla u\|_{1,\gamma} + \lambda_1 \|v\|_1 + \frac{\lambda_2}{2} \|f - u - v\|^2 \right\}.
\]

We observe that the \(\ell^1\)-term in (6.1) dealing with the sparse noise component introduces a further non-differentiability obstacle in the design of a numerical gradient-based optimisation method solving (6.1). Therefore, we further Huber-regularise this term using (4.8) and consider the following optimality

\(^1\)https://www.eecs.berkeley.edu/Research/Projects/CS/vision/bbds/BSB300/html/dataset/images.html
conditions for the regularised problem expressed in a primal-dual form:

\[
\begin{cases}
- \text{div} q - \lambda_2 (f - u - v) = 0, \\
\lambda_1 p - \lambda_2 (f - u - v) = 0, \\
q = \left( \frac{\gamma \nabla u}{\max(1, \gamma |\nabla u|)} \right), \\
p = \left( \frac{\gamma v}{\max(1, \gamma |v|)} \right).
\end{cases}
\]

Starting from an appropriate initial guess for \( u_0 \) and \( v_0 \) (which for our experiments is the noisy image and a sparse vector, respectively), the SSN iteration reads:

\[
\begin{align*}
\delta_q &= - \nabla \delta u + \lambda_2 \delta u + \lambda_2 \delta v = - (\text{div} q + \lambda_2 (f - u - v)), \\
\lambda_1 \delta p + \lambda_2 \delta u + \lambda_2 \delta v &= - (\lambda_1 p - \lambda_2 (f - u - v)), \\
\frac{\nabla q}{\lambda_2 \delta u} &= \frac{\nabla u^T \nabla \delta u}{\lambda_2 \delta u} + \lambda u, \\
\frac{\gamma v}{\lambda_2 \delta v} &= \frac{\nabla v \delta v}{\lambda_2 \delta v} + \gamma v,
\end{align*}
\]

for the increments \( \delta u, \delta v, \delta q \) and \( \delta p \) and where the active sets \( \mathcal{U}_\gamma \) and \( \mathcal{V}_\gamma \) are defined as: \( \mathcal{U}_\gamma := \{ x \in \Omega : \gamma |\nabla u(x)| \geq 1 \} \) and \( \mathcal{V}_\gamma := \{ x \in \Omega : \gamma |v(x)| \geq 1 \} \). As in \([20, 22, 13, 12]\), the SSN iteration above has been modified using the properties of the solution on the final active sets \( \mathcal{U}_\gamma \) and \( \mathcal{V}_\gamma \). Namely, on these sets we have that \( q = \frac{\nabla u}{|\nabla u|} \) and \( p = \frac{v}{|v|} \) together with \( q, p \leq 1 \) a.e. in \( \Omega \). The standard Newton iteration can then be modified accordingly, thus obtaining a positive definite Hessian matrix in each iteration which ensures global convergence.

Figure 4 shows the denoising results for 312 \times 312 pixel images corrupted by a combination of salt & pepper and Gaussian noise of different intensities. The parameters \( \lambda_1 \) and \( \lambda_2 \) have been optimised experimentally with respect to the best Peak Signal to Noise Ratio (PSNR) of the denoised image \( u \) in comparison with the corresponding ground truth \( u_0 \). In all the experiments we observe that the noise is successfully removed from the original image and that is further decomposed into its two constituting salt & pepper and Gaussian components. We observe that the reconstructed images may suffer a loss of contrast resulting in image structures left in the noise components (cf. fifth column of Figure 4). This is a well-known drawback of TV regularisation \([2]\) and can possibly be improved by solving numerically the TV problem using Bregman iteration \([41]\). For improving upon the quality of the reconstruction, higher-order regularisation combined with an IC data discrepancy can be used. This will be the focus of a forthcoming paper \([14]\).

For a closer insight on the noise separation property of the TV-IC model we consider in Figure 5 the classical cameraman image and show the empirical distribution of the impulsive and Gaussian noise components recovered by the TV-IC denoising method. We compare our result with the result obtained by applying the TV-TV^2 regularisation approach proposed in \([42]\) where an higher-order regularisation on the image Hessian of the type

\[
R(u) = \alpha \|\nabla u\|_1 + \beta \|\nabla^2 u\|_1
\]

is considered. The parameters of the model are optimised with respect to the best PSNR value. In both cases, the histograms are in good agreement with our statistical motivation, showing empirical distributions close to the Laplace and Gaussian ones, respectively, but biased because of the use of TV-type regularisation, as studied in \([2]\).

Figure 6 confirms the convergence results of Proposition 5.1, i.e. the single ROF \([43]\) and TV-L^1 model \([26]\) noise model are recovered asymptotically. The salt & pepper and the Gaussian noise component of the model are plotted and their convergence to zero is observed as the corresponding weighting parameter goes to infinity.

In Figure 7 we compare the solutions computed for the TV-IC model \((6.1)\) for different values of \( \lambda_1 \) and \( \lambda_2 \). For comparison, we also present the denoising results computed with the standard denoising models TV-L^1 \([40, 26]\), TV-L^2 \([44]\) and the additive TV-L^1-L^2 combination \([36, 12, 22]\). For reference,
we recall below these models:

\[
\begin{align*}
\min_u \left\{ |Du|_{\gamma}(\Omega) + \lambda_1 \| f - u \|_{L^1(\Omega),\gamma} \right\}, & \quad \text{(TV-L^1)} \\
\min_u \left\{ |Du|_{\gamma}(\Omega) + \frac{\lambda_2}{2} \| f - u \|_{L^2(\Omega)}^2 \right\}, & \quad \text{(TV-L^2)} \\
\min_u \left\{ |Du|_{\gamma}(\Omega) + \lambda_1 \| f - u \|_{L^1(\Omega),\gamma} + \frac{\lambda_2}{2} \| f - u \|_{L^2(\Omega)}^2 \right\}, & \quad \text{(TV-L^1-L^2)}
\end{align*}
\]

We remark that in the following experiments we have also Huber-regularised both the TV term \((4.9)\) and the \(L^1\) term for consistency. As suggested by Proposition \(5.1\), we observe that TV-\(L^1\) and TV-\(L^2\)-type solutions can be obtained from \((6.1)\) by considering large weighting parameters \(\lambda_2\) or \(\lambda_1\), respectively. In these situations, we note that only one component of the noise is smoothed, namely the one corresponding to the active (i.e., non-vanishing) fidelity term in the model. The computed solution of the TV-IC model \((6.1)\) is comparable to the one computed using the TV-\(L^1\)-\(L^2\) denoising model in terms of PSNR values.
Figure 5: Noise separation property of TV-IC denoising model and comparison with higher-order TV-TV^2 denoising model with IC data discrepancy.

1st row: Noisy image (a) corrupted with mixed Gaussian-salt & pepper noise. Density of missing pixels \( d = 10\% \), Gaussian noise variance \( \sigma^2 = 0.005 \), PSNR = 14.37 dB. (b) Ground-truth. (c) TV-IC denoised image, \( \lambda_1 = 235 \) and \( \lambda_2 = 610 \), PSNR = 26.98 dB. (d) TV-TV^2-IC denoised image. \( \alpha = 1, \beta = 0.0023, \lambda_1 = 210, \lambda_2 = 743 \), PSNR = 26.95 dB.

2nd row: Empirical distribution of S&P (e) and Gaussian (f) component for TV-IC.

3rd row: Empirical distribution of S&P (e) and Gaussian (f) component for TV-TV^2-IC.

Because of the bias introduced by TV-type regularisations, the noise components \( v \) and \( f - u - v \) do not follow exact Laplace and Gaussian distributions, respectively, but additionally contain some structures from the image, see [2].

Figure 6: Noise components behaviour for the cameraman example in Figure 5 as parameters \( \lambda_1, \lambda_2 \) of (6.1) go to infinity (logarithmic scale).

(a) \( \| f - u - v \|_2^2 \) decay as \( \lambda_2 \to \infty \)

(b) \( ||v||_1 \) decay as \( \lambda_1 \to \infty \)

but with the additional property of noise decomposition discussed above. Finally, as proved in point (iii) of Proposition 5.1, the noisy image \( f \) is completely recovered by taking large parameters \( \lambda_1 \) and \( \lambda_2 \). For every model considered, all the parameters have been optimised with respect to the best PSNR of the denoised image \( u \). When looking at the asymptotics with respect to one single parameter, one parameter
is set to \( \lambda_1 = 1e7 \) and the other has been optimised with respect to the best PSNR of \( u \). In the case when the joint asymptotics are studied, both parameters have been set to \( \lambda_1 = \lambda_2 = 1e7 \).

\[ \begin{align*}
\text{(a) Noisy image} & \quad \text{(b) TV-}\mathcal{L}^1 & \quad \text{(c) TV-}\mathcal{L}^2 & \quad \text{(d) TV-}\mathcal{L}^1-\mathcal{L}^2 & \quad \text{(e) TV-IC} & \quad \text{(f) TV-IC, } \lambda_2 \gg 1 & \quad \text{(g) TV-IC, } \lambda_1 \gg 1 & \quad \text{(h) TV-IC, } \lambda_1, \lambda_2 \gg 1
\end{align*} \]

Figure 7: Comparison between (TV-\( \mathcal{L}^1 \)), (TV-\( \mathcal{L}^2 \)), (TV-\( \mathcal{L}^1-\mathcal{L}^2 \)) and TV-IC (6.1) reconstructions different values of parameters \( \lambda_1, \lambda_2 \).

1st row: (a) noisy image corrupted with salt & pepper noise (d = 5%) and Gaussian noise with zero mean and variance \( \sigma^2 = 0.005 \), PSNR = 17.18 dB. (b) TV-\( \mathcal{L}^1 \) solution, \( \lambda_1 = 352 \), PSNR = 25.53 dB. (c) TV-\( \mathcal{L}^2 \) solution, \( \lambda_2 = 2121 \), PSNR = 20.26 dB. (d) TV-\( \mathcal{L}^1-\mathcal{L}^2 \) solution, \( \lambda_1 = 351, \lambda_2 = 258 \), PSNR = 25.62 dB.

2nd row: (e) TV-IC solution, \( \lambda_1 = 352, \lambda_2 = 2121 \), PSNR = 26.51 dB. (f) TV-IC solution, \( \lambda_1 = 352, \lambda_2 = 1e7 \), PSNR = 25.61 dB. (g) TV-IC solution, \( \lambda_1 = 1e7, \lambda_2 = 2121 \), PSNR = 20.13 dB. (e) TV-IC solution, \( \lambda_1 = \lambda_2 = 1e7 \), PSNR = 17.17 dB.

6.2 Gaussian-Poisson case: numerical results

For the numerical solution of the mixed Gaussian-Poisson model presented in Section 2.2, we relax the unilateral constraints on \( u \) and \( v \) by adding two standard penalty terms as follows:

\[
\min_{u, v} \left\{ J_\gamma(u,v) := \|\nabla u\|_1, \gamma + \frac{\lambda_1}{2} \|v\|^2 + \lambda_2 \, d_{KL}(f-v,u) + \frac{\gamma_1}{2} \|\min(u,0)\|^2 + \frac{\gamma_2}{2} \|\min(f-v,0)\|^2 \right\}, \quad (6.2)
\]

where \( d_{KL} \) is the discretisation of the Kullback-Leibler functional \( D_{KL} \) in (2.3). In the following numerical experiments, we start from initial values \( \gamma_0^1 = 10 \) and \( \gamma_0^2 = 100 \) and increase them throughout the iterations.

The optimality conditions for (6.2) read:

\[
-\text{div} \left( \frac{\gamma \nabla u}{\max(\gamma,|\nabla u|,1)} \right) + \lambda_2 \left( 1 - \frac{f-v}{u} \right) + \gamma_1 \chi_{\mathcal{S}_u} u = 0,
\]

\[
\lambda_1 \, v - \lambda_2 \, \log \left( \frac{f-v}{u} \right) + \gamma_2 \chi_{\mathcal{S}_v} (v-f) = 0
\]

where \( \chi_{\mathcal{S}_u} \) and \( \chi_{\mathcal{S}_v} \) are the characteristic functions of the sets \( \mathcal{S}_u = \{ x \in \Omega : u(x) < 0 \} \) and \( \mathcal{S}_v = \{ x \in \Omega : v(x) > f(x) \} \), respectively.

Similarly as before, we express the system above in primal-dual form and write the modified SSN
iteration for the increments $\delta_u, \delta_q, \delta_v$ which reads:

$$
\begin{align*}
- \text{div} \delta_q + \lambda_2 \left( \frac{f - v}{u^2} \right) \delta_u + \frac{\lambda_2}{u} \delta_v + \gamma_1 \chi \mathcal{X}_\mu \delta_u = & \text{div} q - \lambda_2 \left( 1 - \frac{f - v}{u} \right) - \gamma_1 \chi \mathcal{X}_\mu u, \\
\delta_q - \frac{\gamma \nabla \delta_u}{\max(1, \gamma |\nabla u|)} + \chi \mathcal{X}_\mu \gamma^2 \frac{\nabla u^T \nabla \delta_u}{\max(1, \gamma |\nabla u|)^2 \max(1, |q|)} = & -q + \frac{\gamma \nabla u}{\max(1, \gamma |\nabla u|)}, \\
\lambda_1 \delta_v + \frac{\lambda_2}{u} \delta_u + \frac{\lambda_2}{f - v} \delta_v + \gamma_2 \chi \mathcal{X}_\mu \delta_v = & - \left( \lambda_1 \frac{v - \lambda_2}{u} \log \left( \frac{f - v}{u} \right) + \gamma_2 \chi \mathcal{X}_\mu (v - f) \right),
\end{align*}
$$

where the set $\mathcal{U}_\gamma$ is the same as the one defined in the previous Section 6.1.

In Figure 8 we report the denoising results for the mixed Gaussian-Poisson model solved via the SSN iteration above. In order to generate the noisy data, we corrupt the image $u^0$ at each pixel $(x_k, y_l)$ of the image domain $\Omega$ with Poisson noise with parameter $u_{k,l}$ and add Gaussian noise with zero mean and different intensities specified in each case. Our results show that the noise components are decomposed as expected, with the Gaussian one being distributed over the whole image domain and the Poisson one depending on the intensity of the image itself. Similarly as above, we observe that some image structures are encoded are captured in the noise components.

![Figure 8](image)

**Figure 8**: **1st column**: Images corrupted with Poisson and Gaussian noise. **2nd column**: Denoising result. **3rd column**: Gaussian noise component. **4th column**: Poisson residuum. **Fifth column**: Poisson noise component.

**1st row**: Gaussian noise variance $\sigma^2 = 0.01$. Noisy image PSNR=19.46 dB. Denoised version PSNR=26.19 dB. Parameters: $\lambda_1 = 2903$, $\lambda_2 = 2107$.

**2nd row**: Gaussian noise variance $\sigma^2 = 0.005$. Noisy image PSNR=22.39 dB. Denoised version PSNR=33.04 dB. Parameters: $\lambda_1 = 2105$, $\lambda_2 = 1896$.

**3rd row**: Gaussian noise variance $\sigma^2 = 0.05$. Noisy image PSNR=18.62 dB. Denoised version PSNR=23.87 dB. Parameters: $\lambda_1 = 809$, $\lambda_2 = 712$.

In Figure 9 we compare the TV-IC reconstruction with the ones obtained using Huber-regularised version of the TV-$L^2$ [44] in $\text{TV-L}^2$, the TV-$KL$ [47, 38], the TV-$L^2+KL$ [22, 12] models as well as
with the TV model combined with exact log-likelihood derived in [33, 34]:

\[
\begin{align*}
&\min_u \left\{ |Du|_{\gamma}(\Omega) + \lambda_2 \int_{\Omega} (u - f \log u) \, dx \right\}, \quad \text{(TV-KL)} \\
&\min_u \left\{ |Du|_{\gamma}(\Omega) + \frac{\lambda_1}{2} \|f - u\|_{L^2(\Omega)}^2 + \lambda_2 \int_{\Omega} (u - f \log u) \, dx \right\}, \quad \text{(TV-}L^2\text{-KL)} \\
&\min_u \left\{ |Du|_{\gamma}(\Omega) - \int_{\Omega} \log \left( \sum_{n=0}^{\infty} \frac{u^n e^{-u}}{n! \sqrt{2\pi\sigma^2}} \right) \, dx \right\}. \quad \text{(TV-GP)}
\end{align*}
\]

Figure 9: Comparison between solutions of TV-IC model (8) and solutions of (TV-}L^2\text{), (TV-KL), (TV-}L^2\text{-KL) and (TV-GP) models. 1st row: (a) noisy image corrupted with Gaussian noise with zero mean and variance \( \sigma^2 = 0.005 \) and Poisson noise with parameter \( u \), PSNR=18.81 dB. (b) TV-}L^2\text{ solution with \( \lambda_1 = 1800, \) PSNR=22.52 dB. (c) TV-KL solution with \( \lambda_2 = 1200, \) PSNR=21.19 dB. 2nd row: (d) TV-}L^2\text{-KL solution with \( \lambda_1 = 520, \lambda_2 = 1100. \) PSNR=22.24 dB. (e) Solution of the TV-GP model (TV-GP) model. PSNR=19.65 dB. (f) Solution of TV-IC model (6.2) with \( \lambda_1 = 1200, \lambda_2 = 1800, \) PSNR=22.97 dB.

The reported results have been optimised in terms of the best PSNR value. We observe that the proposed TV-IC model results in the best reconstruction.

In Figure 10 we report an application of the TV-IC approach for mixed Gaussian-Poisson noise removal for an image acquired by a real camera \(^3\) where a combination of a photon-counting and Gaussian noise can arise for a large value of ISO (see the corresponding database documentation [29]). Compared the solution of the (TV-}L^2\text{) and (TV-KL) models, the TV-IC model outperforms in terms of best PSNR value. For this example, an approximation of the ground truth is provided by the authors by acquiring the same image in a very small noise levels, i.e. setting a very low ISO and with the histogram spanning over the whole dynamic range.

To conclude, we report in Figure 11 numerical tests on the asymptotic behaviour of the model (6.2) as the fidelity weights go to infinity. As shown in Proposition 5.2, both the TV-}L^2\text{ model for Gaussian noise removal (TV-}L^2\text{) (whose solution is denoted by } u_{TV-}L^2\text{) and the TV-KL one (TV-KL) (whose solution is denoted by } u_{TV-}KL\text{) are recovered asymptotically under appropriate norms.}

7 Conclusions and outlook

In this paper we presented a novel variational approach for mixed noise removal. Mixed noise occurs in many applications where different acquisition and/or transmission sources may create interferences of

\(^3\)Images freely available from: [http://fips.fi/photographic_dataset.php](http://fips.fi/photographic_dataset.php)
different statistical nature in the image. Here we focused on two cases of mixed noise, namely salt & pepper noise mixed with Gaussian noise and Gaussian noise combined with Poisson noise. Our variational model, which we call TV-IC, constitutes an infimal convolution combination of standard data fidelities classically associated to one single noise distribution and a total variation (TV) regularisation is used as regularising energy. In a discrete setting, our model is motivated and derived as a joint MAP estimator. The well-posedness of the model is studied in function spaces and by means of standard tools of calculus of variations. Finally, it is shown how single noise models can be recovered from the combined model “asymptotically”, i.e. by letting the weighting parameters of the model become infinitely large. For our numerical experiments we used a semi-smooth Newton (SSN) second-order method to solve efficiently a Huber-regularised version of the problem. In several numerical results and comparisons with existing methods the properties of the proposed TV-IC model are discussed.

Our numerical results show the property of the TV-IC model of decomposing the noise in the data into its single-noise components. Comparisons with state-of-the-art models dealing with the combined case are reported. From a computational point of view, the use of a SSN scheme allows for an efficient computation of the numerical solution of the TV-IC model.

The novel modelling of mixed noise distributions introduced in this paper offers several interesting problems for future research. In particular:

- Parameter learning for TV-IC as outlined in Section 7.1.
- The design of a more general model which could feature a combination of more noise distributions.
• The use of higher-order regularisations such as ICTV [16] and TGV [7]

• The extension to other imaging problems such as cartoon-texture and noise decomposition.

Despite these open problems, we believe that the method presented in this paper is statistically motivated and can be used as an efficient alternative to state-of-the-art data fidelity modelling of mixed noise distributions due to simple numerical realisation and new noise decomposition feature.

7.1 Outlook on parameter learning

Motivated by the recent developments in the context of learning the optimal noise model from examples [22, 13, 12, 35, 23], we report here a preliminary discussion on the selection of the optimal parameters \( \lambda_1 \) and \( \lambda_2 \) for the TV-IC model via a bilevel optimisation approach. Denoting by \( u_{\lambda_1, \lambda_2} \) the image reconstructed using the TV-IC model and by \( \tilde{u} \) the corresponding ground-truth, two standard measures computing the quality of the reconstruction are:

\[
F_{L^2}(u_{\lambda_1, \lambda_2}) := \|u_{\lambda_1, \lambda_2} - \tilde{u}\|_{L^2(\Omega)}^2, \quad F_{L^1\nabla}(u_{\lambda_1, \lambda_2}) := \|D(u_{\lambda_1, \lambda_2} - \tilde{u})\|_{L^1(\Omega)},
\]

where the \( L^1 \) term is Huber-regularised as in (4.8)-(4.9) depending on a parameter \( \gamma \gg 1 \). As motivated in [23], minimising \( F_{L^2} \) corresponds to optimise the SNR of the reconstruction \( u_{\lambda_1, \lambda_2} \), while minimising \( F_{L^1\nabla} \) corresponds to maximise the Structural Similarity Index (SSIM), another standard metric used to assess optimality in imaging applications.

As a preliminary test, we consider here the case of a test image corrupted only with salt & pepper noise with a percentage of missing pixels \( d = 5\% \), see Figure 12. We use the TV-IC model (6.1) solved by SSN method and calculate the value of the Huberised-TV cost functional \( F_{L^1\nabla} \) for different values of the parameters \( \lambda_1 \) and \( \lambda_2 \) and plot in Figure 13 the cost functional as a function of \( \lambda_1 \) and \( \lambda_2 \), drawing a red cross in correspondence of its minimum. For comparison, we repeated the same calculation using the TV-\( L^1-L^2 \) model (TV-\( L^1-L^2 \)) [30, 38]. Whenever the parameters are positive and finite, both models accommodate salt & pepper and Gaussian noise. However, since in the particular case considered only salt & pepper noise is present, we expect in both cases the selected optimal parameters \( \lambda_1^* \) and \( \lambda_2^* \) to enforce a TV-\( L^1 \) type model (TV-\( L^1 \)) which is known to be the appropriate model in the case of pure salt & pepper denoising [40, 26]. Coherently, the plot shows that in both cases the optimal value for \( F_{L^1\nabla} \) is achieved in correspondence with an optimal pair \( (\lambda_1^*, \lambda_2^*) \) enforcing a TV-\( L^1 \) type model, in good agreement with our results in Section 5.1 confirmed numerically in Figure 6.

Figure 12: TV-IC denoising with optimal parameters for pure salt & pepper denoising. For both the TV-IC (6.1) and TV-\( L^1-L^2 \) denoising model (TV-\( L^1-L^2 \)), the optimal parameters \( (\lambda_1^*, \lambda_2^*) \) enforce a TV-\( L^1 \) type model. Figure 12c plots the difference between TV-IC and TV-\( L^1-L^2 \) solutions computed in correspondence with the optimal parameters: the maximum discrepancy between the two has absolute value of \( 4.32 \cdot 10^{-5} \).

Motivated by these preliminary results, we consider the formal treatment of the bilevel learning problem for the selection of optimal parameters \( \lambda_1 \) and \( \lambda_2 \) in the case of a combination of salt & pepper and Gaussian noise. Starting from the lower level Huber-regularised minimisation problem (4.12), similarly in [22], we introduce an elliptic-type regularisation depending on a parameter \( \varepsilon \ll 1 \) and consider the following bilevel problem:

\[
\min_{\lambda_1, \lambda_2 \geq 0} F(u_{\lambda_1, \lambda_2})
\]
subject to:

\[
\min_{u \in H^1_0(\Omega)} \left\{ \varepsilon \| \nabla u \|_{L^2}^2 + |D u|_\gamma(\Omega) + \lambda_1 \|v\|_{L^1(\Omega)} + \frac{\lambda_2}{2} \| f - v \|_{L^2(\Omega)}^2 \right\}.
\]  

(7.3)

Here, we assume the cost functional \( F : H^1_0(\Omega) \to \mathbb{R}^+ \) to be differentiable: examples are the cost functionals in (7.2). Denoting by \((u_\gamma, v_\gamma)\) the optimal solution pair of the lower level problem (7.3) (we stress the dependence on the regularising parameter \( \gamma \)), a necessary and sufficient optimality condition is given by the following system:

\[
\varepsilon(D u_\gamma, D w)_{L^2} + (h_\gamma(D u_\gamma), D w)_{L^2} - \lambda_2(f - v_\gamma - u_\gamma, w)_{L^2} = 0, \quad \text{for all } w \in H^1_0(\Omega),
\]

\[
\lambda_1(h_\gamma(v_\gamma, z)_{L^2} - \lambda_2(f - v_\gamma - u_\gamma, z)_{L^2} = 0 \quad \text{for all } z \in L^2(\Omega).
\]

(7.4a)

(7.4b)

where, similarly to \(22\) Eq.(3.11), by \(h_\gamma(\cdot)\) we have denoted a smoother \(C^2\) Huber-type regularisation of the form

\[
h_\gamma(z) := \begin{cases} \frac{\varepsilon}{\varepsilon_1} (1 - \gamma|z| + \frac{1}{\varepsilon_1}) & \text{if } \gamma|z| - 1 \geq \frac{1}{\varepsilon_1} \\
\frac{\varepsilon}{\varepsilon_1} (1 - \gamma|z| + \frac{1}{\varepsilon_1})^2 & \text{if } \gamma|z| - 1 \in (-\frac{1}{\varepsilon_1}, \frac{1}{\varepsilon_1}) \\
\gamma z & \text{if } \gamma|z| - 1 \leq -\frac{1}{\varepsilon_1} \end{cases}.
\]

Note that in strong form equations (7.4) read:

\[
-\varepsilon \Delta u_\gamma - \text{div} (h_\gamma(D u_\gamma)) - \lambda_2(f - v_\gamma - u_\gamma) = 0 \\
\lambda_1 h_\gamma(v_\gamma) - \lambda_2(f - v_\gamma - u_\gamma) = 0.
\]

We now use the Lagrangian formalism to get an insight in the structure of the optimality system. To do that, we first write (formally) the Lagrangian functional of the problem (7.2) subject to (7.3) as:

\[
\mathcal{L}(\lambda_1, \lambda_2, u, v, p_1, p_2) := F(u) - \varepsilon(Du, Dp_1)_{L^2} - (h_\gamma(Du), Dp_1)_{L^2} + \lambda_2(f - v - u, p_1)_{L^2} - \lambda_1(h_\gamma(v), p_2)_{L^2} + \lambda_2(f - v - u, p_2)_{L^2}
\]

(7.5)

where the function spaces of the adjoint states \(p_1\) and \(p_2\) need to be properly defined. Now, calling \(\Lambda, S\) and \(P\) the first, the second and the third pair of arguments for \(\mathcal{L}\) corresponding to the pair of control, state and adjoint variables, respectively, we have that in correspondence of an optimal solution \((\lambda_1^\gamma, \lambda_2^\gamma, u_\gamma, v_\gamma)\) of the bilevel problem the following two relations hold:

\[
\mathcal{L}_S(\lambda_1^\gamma, \lambda_2^\gamma, u_\gamma, v_\gamma, p_1^\gamma, p_2^\gamma) = 0, \quad (7.6)
\]

\[
\mathcal{L}_\lambda(\lambda_1^\gamma, \lambda_2^\gamma, u_\gamma, v_\gamma, p_1^\gamma, p_2^\gamma)((\alpha, \beta)^T - (\lambda_1^\gamma, \lambda_2^\gamma)^T) \geq 0, \quad \text{for every } \alpha, \beta \geq 0. \quad (7.7)
\]
The computation of the derivatives renders:

\[
\begin{align*}
\frac{\partial L}{\partial u}(\lambda_1^1, \lambda_2^2, u_\gamma, v_\gamma, p_1^1, p_2^2)[w_1] &= (F'(u_\gamma), w_1)_{L^2} - \varepsilon(Dp_1^1, Dw_1)_{L^2} - (h'_\gamma(Du_\gamma) Dw_1, Dp_1^1)_{L^2} \\
&\quad - \lambda_2(p_1^1, w_1)_{L^2} - \lambda_2(p_2^1, w_1)_{L^2} = 0, \quad \text{for all } w_1 \in H^1_0(\Omega), \\
\frac{\partial L}{\partial v}(\lambda_1^1, \lambda_2^2, u_\gamma, v_\gamma, p_1^1, p_2^2)[w_2] &= -\lambda_2(p_1^2, w_2)_{L^2} - \lambda_1(p_2^2, h'_\gamma(v)w_2)_{L^2} \\
&\quad - \lambda_2(p_2^2, w_2)_{L^2} = 0, \quad \text{for all } w_2 \in L^2(\Omega),
\end{align*}
\]

which be rewritten as:

\[
\begin{align*}
\varepsilon(Dp_1^1, Dw_1)_{L^2} + (h'_\gamma(Du_\gamma)^*Dp_1^1, Dw_1)_{L^2} + \\
&- \lambda_2(p_1^1, w_1)_{L^2} = (F'(u_\gamma), w_1)_{L^2}, \quad \text{for all } w_1 \in H^1_0(\Omega), \\
&- \lambda_1 h'_\gamma(v)^*p_2^2 + \lambda_2(p_1^1 + p_2^2) = 0 \quad \text{a.e. in } \Omega.
\end{align*}
\]

Therefore, the optimality condition (7.7) reads:

\[
\begin{align*}
\frac{\partial L}{\partial \lambda_1}(\lambda_1^1, \lambda_2^2, u_\gamma, v_\gamma, p_1^1, p_2^2)(\alpha - \lambda_1^1) &= \left( \int_\Omega h'_\gamma(v_\gamma)p_2^2 \right) (\lambda_1^1 - \alpha) \geq 0, \\
\frac{\partial L}{\partial \lambda_2}(\lambda_1^1, \lambda_2^2, u_\gamma, v_\gamma, p_1^1, p_2^2)(\beta - \lambda_2^2) &= \left( \int_\Omega (f - v_\gamma - u_\gamma)(p_1^1 + p_2^2) \right) (\lambda_2^2 - \beta) \geq 0
\end{align*}
\]

for every \( \alpha, \beta \geq 0 \).

Therefore, for an optimal quadruplet \((\lambda_1^1, \lambda_2^2, u_\gamma, v_\gamma) \in \mathbb{R}^+ \times \mathbb{R}^+ \times H^1_0(\Omega) \times L^2(\Omega)\) there must exist \((p_1^1, p_2^2) \in H^1_0(\Omega) \times L^2(\Omega)\) such that the following optimality system holds (in strong form):

\[
\begin{align*}
-\varepsilon \Delta u_\gamma - \text{div } h'_\gamma(Du_\gamma) - \lambda_2(f - v_\gamma - u_\gamma) = 0 & \quad \text{a.e. in } \Omega \\
\lambda_1 h'_\gamma(v_\gamma) - \lambda_2(f - v_\gamma - u_\gamma) = 0 & \quad \text{a.e. in } \Omega \\
-\varepsilon \Delta p_1^1 - \text{div } \left( h'_\gamma(Du_\gamma)^*Dp_1^1 \right) + \lambda_2(p_1^1 + p_2^2) = F'(u_\gamma) & \quad \text{a.e. in } \Omega \\
u_\gamma = 0, & \quad p_1^1 = 0 \quad \text{on } \Gamma \\
\lambda_1 h'_\gamma(v_\gamma)^*p_2^2 - \lambda_2(p_1^1 + p_2^2) = 0 & \quad \text{a.e. in } \Omega \\
m_1 := -\int_\Omega h'_\gamma(v_\gamma)p_2^2, & \quad m_2 := \int_\Omega (f - v_\gamma - u_\gamma)(p_1^1 + p_2^2) \\
\lambda_1^1 \geq 0, & \quad \mu_1 \geq 0, & \quad \mu_1 \lambda_1^1 = 0 \quad \text{for } i = 1, 2.
\end{align*}
\]

**Remark 7.1.** For the Gaussian and Poisson case described in Section 2.2 additional difficulties have to be taken into account. Namely, in this case the admissible sets \( \mathcal{A} \) and \( \mathcal{B} \) in (2.5) are required to guarantee convexity of the data discrepancy \( \Phi \) in the variable \( u \). Since this involves point-wise bounds on a state variable, the existence of Lagrange multipliers (even of low regularity) has to be carefully justified and the numerical solution of the problem becomes challenging.

The rigorous study of the optimality system derived above and its limit as \( \gamma \to \infty \) are an additional interesting topic of future research.

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**References**


